

NON-BIG SUBGROUPS FOR l LARGE

THOMAS BARNET-LAMB

ABSTRACT. Lifting theorems form an important collection of tools in showing that Galois representations are associated to automorphic forms. (Key examples in dimension $n > 2$ are the lifting theorems of Clozel, Harris and Taylor and of Geraghty.) All present lifting theorems for $n > 2$ dimensional representations have a certain rather technical hypothesis—the residual image must be ‘big’. The aim of this paper is to demystify this condition somewhat.

For a fixed integer n , and a prime l larger than a constant depending on n , we show that n dimensional mod l representations which fail to be big must be of one of three kinds: they either fail to be absolutely irreducible, are induced from representations of larger fields, or can be written as a tensor product including a factor which is the reduction of an Artin representation in characteristic zero. Hopefully this characterization will make the bigness condition more comprehensible, at least for large l .

CONTENTS

1. Introduction	1
2. Big image	4
3. Group theory	5
4. Sturdy subgroups	7
5. The main result	21
References	30

1. INTRODUCTION

1.1. Recent years have seen some progress in proving modularity lifting theorems and potential modularity theorems for Galois representations of dimension $n > 2$: for instance, see [CHT08, HSBT06, Ger09, BLGHT09, BLGG09, BLGGT10]. One hypothesis in common to all these theorems is that the image of the residual representation to which we want to apply the theorem must be ‘big’, a rather technical hypothesis first introduced by Clozel, Harris and Taylor in [CHT08]. (See the beginning of §2 for a full statement of this condition.) In fact, several of these theorems rely on a stronger hypothesis, introduced in [BLGHT09]: the image of the residual representation must be ‘ M -big’. (Again, see §2 for a full statement of this condition. We remark that generally one writes ‘ m -big’ rather than ‘ M -big’, but we suffer from a shortage of letters in this paper and have had to economize.)

While on the one hand it seemed to be fairly easy to show that these bigness and M -bigness hypotheses hold with a specific example in mind (see, for instance Corollaries 2.4.3 and 2.5.4 and Lemmas 2.5.5 and 2.5.6 of [CHT08] or Lemmas 7.3–4

of [BLGHT09]), it remained for some time the case that, considered in general, the condition was rather mysterious—especially given the fact that its precise definition is so technically involved.

A considerable advance in the situation came with the paper [SW10]. The main result of this paper was to investigate the question of bigness in compatible systems, showing that for compatible systems of Galois representations which are ‘strongly irreducible’ (in the sense that for any given finite field extension, the restrictions of the representations in the compatible system to that extension are irreducible at almost all places), the residual image will be big at a positive Dirichlet density of places. Along the way, they prove that a large class of subgroups are big (in particular, the images of certain algebraic representations), thus giving a rather general situation in which bigness can be deduced (for instance, the specific examples considered in [CHT08, BLGHT09], are much more specific and are indeed immediate consequences). This was generalized to M -bigness by White in [Whi10].

The aim of this paper is to push the powerful techniques introduced by Snowden and Wiles in that paper a little further, combining them with a little group theory to give a still larger class of representations with big image. The class is rather broad, and indeed the only representations *not* in the class are of rather special kinds.

Our main theorem is the following (see Theorem 5.1.3):

Theorem 1.1.1. *For each pair of positive integers n and M , there is an integer $C(M, n)$ with the following property. Let $l > C(M, n)$ be a prime, k/\mathbb{F}_l a finite extension with $l \nmid [k : \mathbb{F}_l]$, Γ_0 a profinite group, and $r : \Gamma_0 \rightarrow \mathrm{GL}_n(k)$ a representation. For convenience of notation, let us choose a number field L and prime λ of L such that $\mathcal{O}_L/\lambda\mathcal{O}_L = k$. Suppose that the image of r is not M -big. Then one of the following must hold:*

- (1) *r does not act absolutely irreducibly on k^n ,*
- (2) *there is a proper subgroup $\Gamma'_0 < \Gamma_0$, an integer $m|n$, and a representation $r' : \Gamma'_0 \rightarrow \mathrm{GL}_m(k)$ such that $r = \mathrm{Ind}_{\Gamma'_0}^{\Gamma_0} r'$, or*
- (3) *there are representations $r_1 : \Gamma_0 \rightarrow \mathrm{GL}_m(\mathcal{O}_L)$ and $r_2 : \Gamma_0 \rightarrow \mathrm{GL}_{m'}(k)$, with open kernels and with $m > 1$ and $n = mm'$, such that $r = \bar{r}_1 \otimes r_2$.*

Furthermore, in case (3) the order of the image of r_1 can be bounded in terms of n .

Remark 1.1.2. It is natural to ask whether the constant $C(m, n)$ is effective. The answer is that, in principle, a sufficiently assiduous study of the proofs here and in the various papers cited here should allow one to determine effective bounds on $C(m, n)$. (The same holds for the similar constants $D(n, N, a)$ and $E(n, m, a)$ below.) On the other hand, one would expect these bounds to be rather large, and so we will not attempt this arduous calculation here.

Remark 1.1.3. The condition that $l \nmid [k : \mathbb{F}_l]$ is probably harmless, but in case it were ever problematic, it is worth mentioning that it is effectively dispensable. In particular, one of the requirements of a subgroup $\Gamma < \mathrm{GL}_n(k)$ being (M) -big is that it have no l power order quotient. In all applications we know of, however, it suffices to have the weaker property that any normal subgroup Γ' of l -power index still satisfies all the other properties defining (M) -bigness, and one could imagine replacing the definition of bigness with one only demanding this weaker property. If we were to make this change in the definition, the condition that $l \nmid [k : \mathbb{F}_l]$ in the theorem could be removed.

In applications, one will often be working with crystalline Galois representations which have regular Hodge-Tate numbers, and one may have the flexibility to choose l large compared to those Hodge-Tate numbers. (For instance, one might be working with a compatible family of Galois representations.) In this case, we may suppress the third alternative in Theorem 1.1.1.

Lemma 1.1.4. *Suppose n and N are positive integers, F is a number field and that for each embedding $\tau : F \hookrightarrow \mathbb{C}$, we have chosen a set a_τ of n distinct integers. Then there exists a number $D(n, N, a)$ with the following property. Whenever:*

- $l > D(n, N, a)$ is a rational prime which is unramified in F ,
- $L \subset \overline{\mathbb{Q}}_l$ is a finite extension of \mathbb{Q}_l (with ring of integers \mathcal{O}_L , residue field k),
- $\rho : G_F \rightarrow \mathrm{GL}_n(\mathcal{O}_L)$ is a crystalline Galois representation, and
- $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ is an isomorphism such for each embedding $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_l$ that the multiset of Hodge-Tate numbers of ρ with respect to it is precisely the set $a_{\iota \circ \sigma}$ (and in particular, the Hodge-Tate numbers are distinct),

then we can never find an integer $m > 1$ and some Artin representation $\rho' : G_F \rightarrow \mathrm{GL}_m(\mathcal{O}_L)$ with image of order at most N , and a mod l representation $r'' : G_F \rightarrow \mathrm{GL}_{m'}(k)$, such that $\bar{\rho} \cong \rho' \otimes r''$.

Corollary 1.1.5. *Suppose n and M are positive integers, F is a number field and that for each embedding $\sigma : G \hookrightarrow \overline{\mathbb{Q}}_l$, a_σ is a set of n distinct integers. Then there exists a number $E(n, M, a)$ with the following property. Whenever:*

- $l > E(n, M, a)$ is a rational prime which is unramified in F ,
- $L \subset \overline{\mathbb{Q}}_l$ is a finite extension of \mathbb{Q}_l (ring of integers \mathcal{O}_L , residue field k),
- $\rho : G_F \rightarrow \mathrm{GL}_n(\mathcal{O}_L)$ is a crystalline Galois representation, and
- $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ is an isomorphism such for each embedding $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_l$ that the multiset of Hodge-Tate numbers of ρ with respect to it is precisely the set $a_{\iota \circ \sigma}$ (and in particular, the Hodge-Tate numbers are distinct),

and we moreover have that $\bar{\rho}(G_F)$ fails to be an M -big subgroup of $\mathrm{GL}_n(k)$, then we may conclude that one of the following holds:

- (1) $\bar{\rho}$ does not act absolutely irreducibly on k^n , or
- (2) there is a proper subgroup $G' < G_F$ and representation $r' : G' \rightarrow \mathrm{GL}_m(k)$ such that $\bar{\rho} = \mathrm{Ind}_{G'}^{G_F} r'$.

(The lemma is proved at the end of §5.1; the corollary is then immediate given the theorem.)

We now say something about the methods used to prove our main result above. There are three main ingredients. The first is, as we have already mentioned, the ideas of [SW10] (and their generalizations in [Whi10]). We are unable to directly apply their results, and instead will have to look ‘under the hood’ a little, recapitulating some arguments from their paper in slightly modified settings. (See §4.) The second is the Aschbacher-Dynkin theorem, a classification of the maximal subgroups of $\mathrm{GL}_n(k)$ not containing $\mathrm{SL}_n(k)$ (here k/\mathbb{F}_l is a finite extension). We prove our main theorem by applying the Aschbacher-Dynkin theorem to the image of the representation in question, and then breaking into cases according to what kind of maximal subgroup the image lies in. In most of these cases, we are either done immediately or can reduce to a problem concerning a smaller-dimensional

representation for which we may inductively assume our result holds, and from this deduce the result for the original representation.

The remaining case is where the image of our subgroup in $\mathrm{GL}_n(k)/k^\times$ is an almost simple group. In this case, we apply our third ingredient: results of Larsen which provide good control over which almost simple groups might arise in such a situation. In particular, they are either drawn from a finite set of groups depending only on n and not on l (in this case, we can arrange for l to be large enough that such a representation lifts to characteristic zero), or are groups of Lie type in characteristic l (in which case the image of our representation is essentially the image of an algebraic representation, and we can finish our argument by appeal to the ideas of Snowden and Wiles we have already mentioned.)

Of course, there are various wrinkles—most notably, our inductive argument actually proves something slightly stronger than the main theorem, so that our inductive hypothesis is strong enough when we need to use it.

We now explain the organization of the rest of this paper. In section 2, we recall the definitions of ‘big’ and ‘ M -big’, and also some elementary properties of bigness proved by Snowden and Wiles (and also their analogues for M -bigness, which are due to White in [Whi10]). In section 3 we state the various group-theoretical tools we will be using, most notably stating the Aschbacher-Dynkin theorem in the form we will use it, and recalling the results of Larsen mentioned above. In section 4 we introduce the notion of a ‘sturdy’ subgroup of $\mathrm{GL}_n(k)$, for k/\mathbb{F}_l finite, showing that sturdy subgroups are big and have various desirable properties. The notion of ‘sturdiness’ is very closely related to the notion of ‘being the image of an algebraic representation’, and our arguments here draw heavily on [SW10, Whi10]. Finally, in section 5 we combine these results to prove the main theorem, and close the section with some (very vague) comments about the degree to which we expect representations which satisfy one of the properties (1)–(3) in Theorem 1.1.1 will nonetheless have big image.

Acknowledgements. It will be clear to the reader how much this work depends on the ideas of Andrew Snowden and Andrew Wiles, and it is a pleasure to acknowledge my debt to them. I also thank them for keeping me updated on the progress of revisions to [SW10], and for encouraging me to press ahead in proving the results in this paper. Finally, I am grateful to Toby Gee and Andrew Snowden for providing very helpful comments on an earlier draft of this paper.

2. BIG IMAGE

2.1. We begin by recalling the definition of M -big from [BLGHT09] (see Definition 7.2 there).

Definition 2.1.1. Let k/\mathbb{F}_l be algebraic and m a positive integer. We say that a subgroup $H \subset \mathrm{GL}_n(k)$ of $\mathrm{GL}_n(k)$ is M -big if the following conditions are satisfied.

- H has no l -power order quotient.
- $H^0(H, \mathfrak{sl}_n(k)) = (0)$.
- $H^1(H, \mathfrak{sl}_n(k)) = (0)$.
- For all irreducible $k[H]$ -submodules W of $\mathfrak{gl}_n(k)$ we can find $h \in H$ and $\alpha \in k$ such that:
 - α is a simple root of the characteristic polynomial of h , and if β is any other root then $\alpha^M \neq \beta^M$.

- Let $\pi_{h,\alpha}$ (respectively $i_{h,\alpha}$) denote the h -equivariant projection from k^n to the α -eigenspace of h (respectively the h -equivariant injection from the α -eigenspace of h to k^n). Then $\pi_{h,\alpha} \circ W \circ i_{h,\alpha} \neq 0$.

We will use ‘big’ as a synonym for ‘1-big’. This is consistent with the definition of ‘big’ in [CHT08].

The following lemmata establish basic properties of bigness and M -bigness which will be constantly of use to us. They were essentially proved in [SW10]. (The statements in that paper are in the context of bigness, but the proofs trivially extend to M -bigness, as was noted in [Whi10]—see section 2 of that paper.)

Proposition 2.1.2. *Let k/\mathbb{F}_l be algebraic, and let M and n be positive integers. If $G < \mathrm{GL}_n(k)$ has a normal subgroup H , of index prime to l , which is M -big, then G is M -big.*

Proposition 2.1.3. *Let k/\mathbb{F}_l be algebraic, and let M and n be positive integers. Suppose $G < \mathrm{GL}_n(k)$. Then G is M -big if and only if $k^\times G$ is.*

3. GROUP THEORY

3.1. We remind ourselves of a simple form of the Aschbacher-Dynkin theorem, mostly to fix notation for what follows. The theorem is due independently to Aschbacher and Dynkin, and the version we use here is somewhat less powerful than the full version in the original (independent) arguments of Aschbacher (see [Asc84]) and Dynkin (see [Dyn52] for the original paper in Russian, or [Dyn00] for an English translation). See [Wil09, §3.10.3] for the theorem proved in the precise form we shall use it. Let us first define certain subgroups of the general linear groups over a finite field.

Definition 3.1.1. Let l be a prime, k/\mathbb{F}_l a finite extension. Let us describe certain subgroups of $\mathrm{GL}_n(k)$.

- (1) For each linear subspace $V \subset k^n$ of dimension d , $0 < d < n$, we have a subgroup G_V which sends V onto itself. We have

$$G_V \cong (l^{[k:\mathbb{F}_l]d(n-d)}) \rtimes (\mathrm{GL}_d(k) \times \mathrm{GL}_{n-d}(k))$$

(where $(l^{[k:\mathbb{F}_l]d(n-d)})$ denotes some group of order $l^{[k:\mathbb{F}_l]d(n-d)}$). (For obvious reasons, we call G_V a *reducible* subgroup.)

- (2) For each direct sum decomposition $k^n = V_1 \oplus \cdots \oplus V_m$, where $m|n$ and each V_i has dimension n/m , we have a subgroup $G_{V_1 \oplus \cdots \oplus V_m}$ which stabilizes the direct sum decomposition (but need not stabilize the individual summands in the direct sum). We have $G_{V_1 \oplus \cdots \oplus V_m} \cong \mathrm{GL}_{n/m}(k) \wr S_m$ and we call $G_{V_1 \oplus \cdots \oplus V_m}$ an *imprimitive* subgroup.
- (3) For each decomposition $k^n \equiv V_1 \otimes V_2$, where $n = \dim V_1 \dim V_2$, we have a subgroup $G_{V_1 \otimes V_2}$ stabilizing the tensor product decomposition. We have $G_{V_1 \otimes V_2} \cong \mathrm{GL}_{n_1}(k) \circ \mathrm{GL}_{n_2}(k)$, where $n_i = \dim V_i$ and \circ denotes the central product.
- (4) For each decomposition $k^n \equiv V_1 \otimes \cdots \otimes V_m$, where the V_i all have the same dimension (d say), and so $d^m = n$, we have a subgroup $G_{V_1 \otimes \cdots \otimes V_m}$ of transformations which preserve this tensor product decomposition (but which may rearrange the factors amongst themselves). We have

$$k^\times G_{V_1 \otimes \cdots \otimes V_m} / k^\times \cong \mathrm{PGL}_d(k) \wr S_m.$$

- (5) If $n = p^m$ for some odd prime p , then we have certain subgroups of $\mathrm{GL}_n(\bar{\mathbb{Q}})$ isomorphic to the ‘extraspecial’ group p_+^{1+2m} , whose normalizers in $\mathrm{GL}_n(\bar{\mathbb{Q}})$ are isomorphic to $p_+^{1+2m} \rtimes \mathrm{Sp}_{2m}(\mathbb{F}_p)$. (See §3.10.2 of [Wil09].) Assuming our finite field k is large enough, these subgroups reduce mod k to give a subgroup of $\mathrm{GL}_n(k)$. We can form a larger subgroup, $G_{p^{1+2m}}$, by additionally including all central elements, and taking subgroup this generates.

If $n = 2^m$ for some m , we have similar subgroups 2_ϵ^{1+2m} of $\mathrm{GL}_n(\bar{\mathbb{Q}})$ where $\epsilon \in \{+, -\}$, whose normalizers are $2_\epsilon^{1+2m} \rtimes \mathrm{GO}_{2m}^\epsilon(\mathbb{F}_2)$, and again, if k is large enough, these subgroups reduce mod k to give a subgroup of $\mathrm{GL}_n(k)$. We can form a larger subgroup, $G_{2_\epsilon^{1+2m}}$, by additionally including all central elements, and taking subgroup this generates.

We call $G_{p^{1+2m}}$ and $G_{2_\epsilon^{1+2m}}$ *subgroups of extraspecial type*.

- (6) Let H be an almost simple group (a group H satisfying $G < H < \mathrm{Aut} G$ for some simple group G). Let $\bar{r} : H \rightarrow \mathrm{PGL}_n(k)$ be an irreducible projective modular representation. Then we can form a subgroup $G_{\bar{r}} := \bar{r}(H)k^\times$.

Theorem 3.1.2 (Aschbacher, Dynkin). *Let l be a prime, k/\mathbb{F}_l a finite extension. Let G be a subgroup of $\mathrm{GL}_n(k)$ which does not contain $\mathrm{SL}_n(k)$. Then G is contained in one of the subgroups of the forms listed in Definition 3.1.1.*

3.2. We also recall the following result of Larsen, which is essentially a combination of [Lar95a, Lemmas 1.5–1.8].

Proposition 3.2.1 (Larsen). *Fix a semisimple group scheme $\mathcal{G}/\mathbb{Z}[1/N]$. Then there is an integer A and a finite collection of finite simple groups S , such that for all $l > A$, all finite extensions k/\mathbb{F}_l , and all subgroups $H < \mathcal{G}(k)$ which are nonabelian finite simple groups, we have:*

- (1) H is isomorphic to some member of S , or
- (2) H is a derived group of an adjoint group of Lie type, $H \cong \mathcal{D}(\mathcal{H}(\mathbb{F}_q))$, \mathcal{H} simple, which moreover satisfies $l|q$ and $\mathcal{D}(\mathcal{H}(\mathbb{F}_q)) = \mathrm{Im}(\mathcal{H}^{\mathrm{sc}}(\mathbb{F}_q) \rightarrow \mathcal{H}(\mathbb{F}_q))$ (where $\mathcal{H}^{\mathrm{sc}}$ denotes the simply connected cover of \mathcal{H}).

Proof. If $k = \mathbb{F}_l$, this follows immediately from [Lar95a, Lemmas 1.5–1.8], together with the classification of finite simple groups (as given, say, in [Lar95a, §1.3]). To see that the proposition holds in the more general case we give here, first note that the proof of [Lar95a, Lemma 1.4] actually gives a more general result, where we replace \mathbb{F}_l with any finite extension k/\mathbb{F}_l . Then Lemmas 1.5–1.8 have similar generalizations, again with essentially unaltered proofs. The proposition follows. The fact that we may take $\mathcal{D}(\mathcal{H}(\mathbb{F}_q)) = \mathrm{Im}(\mathcal{H}^{\mathrm{sc}}(\mathbb{F}_q) \rightarrow \mathcal{H}(\mathbb{F}_q))$ is arranged by including the finitely many finite simple groups of Lie type for which this is not true in S . \square

3.3. Finally, we rehearse some of the standard theory which relates, for a Chevalley group over a finite field of characteristic l , modular representations (in characteristic l) of the Chevalley group considered as an algebraic group with algebraic representations of the Chevalley group; and theory which allows us to lift these algebraic representations to characteristic zero under certain circumstances.

In particular, we have on the one hand the following result of Steinberg (see [Ste63, 1.3] and [Ste68, 13.3]):

Theorem 3.3.1. *Suppose that k/\mathbb{F}_l is a finite extension, and \mathcal{G}/k is an almost simple, simply connected algebraic group for which we have made a choice of a maximal torus and a system of simple roots. Then there is a bijection between irreducible projective \mathbb{F}_l representations of the abstract group $\mathcal{G}(k)$ and tensor products:*

$$\bigotimes_{i=0}^{[k:\mathbb{F}_l]-1} V_i^{\text{Frob}^i}$$

where each V_i is an irreducible algebraic representation with highest weight λ satisfying $0 \leq \lambda(\mathbf{a}) \leq l-1$ for each simple root \mathbf{a} , and where the superscript ‘Frob’ indicates precomposing a representation (considered as a map $\mathcal{G}(k) \rightarrow \text{PGL}_n(\mathbb{F}_l)$ for some n) with the i th power of the Frobenius map $\mathcal{G}(k) \rightarrow \mathcal{G}(k)$. The bijection is given by mapping such tensor products into abstract group representations by restricting them to k points in the obvious way.

And, on the other, the following result which is essentially due to Larsen:

Theorem 3.3.2. *Suppose that we are in the situation of the previous theorem, and that \mathcal{G} is split over k and $l \geq 3(\dim V_i)^2$. Then each of the representations V_i lifts to characteristic zero, in the sense that there is some algebraic group \mathfrak{G}_i over $W(k)$ and algebraic representation \tilde{V}_i of \mathfrak{G}_i , such that base changing back to k we recover \mathcal{G} and V_i .*

Proof. The proof of Proposition 4.4 of [Lar95b] extends to give us what we need. See also [Jan97]. \square

We will record a trivial corollary of this second theorem, which will be very important to us.

Corollary 3.3.3. *Suppose n is a positive integer. Then there are constants $C_0(n)$ and $C_1(n)$ with the following property. Suppose that $l > C_0(n)$ is a prime, that k/\mathbb{F}_l is a finite extension, that \mathcal{G}/k is an almost simple, simply connected algebraic group, and that we are given an irreducible \mathbb{F}_l representation of the abstract group $\mathcal{G}(k)$, at most n dimensional, which corresponds to*

$$\bigotimes_{i=0}^{[k:\mathbb{F}_l]-1} V_i^{\text{Frob}^i}$$

under the correspondence of Theorem 3.3.1. Then the norms $\|V_i\|$ of the representations V_i (see §3.2 of [SW10]) satisfy $\|V_i\| < C_1(n)$.

Proof. This follows straightforwardly from the fact that the V_i lift to characteristic 0; see Proposition 3.5 of [SW10]. \square

4. STURDY SUBGROUPS

4.1. In this section, we will define the notion of a *sturdy* subgroup of $\text{GL}_n(k)$; the point of introducing this notion is that on the one hand, being sturdy suffices to be M -big, at least for l large; and on the other hand, being sturdy behaves well under ‘taking tensor products’ (see Lemma 4.2.1). Sturdy subgroups are very closely related to the images of algebraic representations, and as such, this section will draw heavily on the work of Snowden and Wiles discussed above. One wrinkle, however, is that we will need to deal with representations which are tensor products of algebraic representations defined over different fields.

Definition 4.1.1. Suppose l is a prime number, n an integer, k/\mathbb{F}_l a finite extension, and $\Gamma < \mathrm{GL}_n(k)$ a subgroup. We say that Γ is *very sturdy* if it contains k^\times , acts absolutely irreducibly, and we can find:

- a sequence k_1, \dots, k_m of fields, each of which is an intermediate field between k and \mathbb{F}_l ,
- for each i , $1 = 1, \dots, m$, an almost simple, connected, simply connected algebraic group \mathcal{G}_i^{sc}/k_i , which remains almost simple when we base change to \bar{k} , and
- for each i , $1 = 1, \dots, m$, a faithful projective representation over k of the abstract group $\pi_i(\mathcal{G}_i^{sc}(k_i))$

$$r_i : \pi_i(\mathcal{G}_i^{sc}(k_i)) \rightarrow \mathrm{PGL}_{n_i}(k),$$

where

- \mathcal{G}_i/k_i is the algebraic group we get by taking the quotient of \mathcal{G}_i^{sc}/k_i by its center (which will be its unique maximal normal subgroup), and
- $\pi_i : \mathcal{G}_i^{sc} \rightarrow \mathcal{G}_i$ is the covering map, which we also consider as a map of k_i points,

(so that $\pi_i(\mathcal{G}_i^{sc}(k_i))$ is a finite simple group of Lie type)

such that:

- the abstract group $\pi_i(\mathcal{G}_i^{sc}(k_i))$ is in fact a simple group for all i ,
- $n = \prod_i n_i$ (this will follow automatically from the other conditions, but we mention it to orient the reader), and
- Γ/k^\times is conjugate to the subgroup of $\mathrm{PGL}_n(k)$ given by the image of the map

$$\prod_i \pi_i(\mathcal{G}_i^{sc}(k_i)) \xrightarrow{\prod_i r_i} \prod_i \mathrm{PGL}_{n_i}(k) \hookrightarrow \mathrm{PGL}_n(k).$$

Definition 4.1.2. Suppose l is a prime number, k/\mathbb{F}_l a finite extension, and $\Gamma < \mathrm{GL}_n(k)$ a subgroup. We say that Γ is *sturdy* if there is a chain $\Gamma_1 \triangleleft \Gamma_2 \triangleleft \dots \triangleleft \Gamma_r = \Gamma$ of subgroups of Γ , each normal in the next, such that:

- (1) for each i , $1 \leq i \leq r-1$, $(\Gamma_{i+1} : \Gamma_i)$ is coprime to l , and
- (2) Γ_1 is very sturdy, in the sense of Definition 4.1.1.

Here are an easy observation and a remark:

Lemma 4.1.3. Suppose l is a prime number, n an integer, k/\mathbb{F}_l a finite extension, and $\Gamma < \mathrm{GL}_n(k)$ a very sturdy subgroup. Then, as an abstract group Γ/k^\times is a product of the simple groups $\pi_i(\mathcal{G}_i^{sc}(k_i))$, which are simple groups of Lie type in characteristic l .

Remark 4.1.4. If Γ is a very sturdy subgroup, then we have the freedom to replace any of the groups \mathcal{G}_i^{sc} referred to in Definition 4.1.1 with other groups which yield an isomorphic $\pi_i(\mathcal{G}_i^{sc}(k_i))$ (and then modify the corresponding r_i by composing with the isomorphism between the old and new $\pi_i(\mathcal{G}_i^{sc}(k_i))$ s), and the new \mathcal{G}_i^{sc} s will still satisfy the properties required to show that Γ is very sturdy. As explained in [CCN⁺, §2.4–1, §3.2], for any finite simple group of Lie type in characteristic l (that is, any group which could arise amongst the $\pi_i(\mathcal{G}_i^{sc}(k_i))$ s above), it is possible to find a \mathcal{G}^{sc} which gives rise to it as above such that the algebraic group \mathcal{G}_i^{sc} in addition enjoys the following property: it has a Borel subgroup \mathcal{B} and torus \mathcal{T} , such that the Borel and torus are defined over k and hence sent onto themselves by

$\text{Gal}(\bar{k}/k)$. A generator of $\text{Gal}(\bar{k}/k)$ acts as a graph automorphism of the Dynkin diagram of \mathcal{G}^{sc} , and so the simple roots can be arranged in cycles under this action. We will from time to time assume that the \mathcal{G}_i^{sc} have been chosen with these special properties. In this case, we will write \mathcal{T}_i for the torus in \mathcal{G}_i according to the properties just described. Thus, in particular the finite group $\mathcal{T}(k_i)$ is a product of \mathbb{G}_m s of extension fields of k_i , one for each orbit of the simple roots under Frobenius, where for each orbit the extension field has degree over k_i equal to the number of roots in the orbit.

4.2. Properties of sturdy subgroups.

Lemma 4.2.1. *Suppose that n_1, n_2 are positive integers, and we write $n = n_1 n_2$. Suppose we have chosen an isomorphism $k^n \cong k^{n_1} \otimes k^{n_2}$; this then gives us a natural map $\text{PGL}_{n_1}(k) \times \text{PGL}_{n_2}(k) \hookrightarrow \text{PGL}_n(k)$, whose image is the collection of elements of $\text{PGL}_n(k)$ preserving the tensor product decomposition. Suppose that $\Gamma < \text{PGL}_n(k)$ is a subgroup, which is contained in the image of $\text{PGL}_{n_1}(k) \times \text{PGL}_{n_2}(k) \hookrightarrow \text{PGL}_n(k)$, so we can think of Γ as a subgroup of $\text{PGL}_{n_1}(k) \times \text{PGL}_{n_2}(k)$. Let $\pi_1 : \text{PGL}_{n_1}(k) \times \text{PGL}_{n_2}(k) \rightarrow \text{PGL}_{n_1}(k)$ denote the projection map, and similarly for π_2 . Let $q : \text{GL}_n(k) \twoheadrightarrow \text{PGL}_n(k)$ denote the natural quotient map, and similarly q_1 and q_2 . Suppose that $q_1^{-1}(\pi_1(\Gamma))$ and $q_2^{-1}(\pi_2(\Gamma))$ are both sturdy. Then $q^{-1}(\Gamma)$ is sturdy if it acts absolutely irreducibly.*

Proof. We shall write $\Gamma^{(1)}$ for $\pi_1(\Gamma)$ and $\Gamma^{(2)}$ for $\pi_2(\Gamma)$. Goursat's Lemma tells us that we can find a group G and surjections $p^{(1)} : \Gamma^{(1)} \rightarrow G$, $p^{(2)} : \Gamma^{(2)} \rightarrow G$, such that Γ , considered as a subgroup of $\text{PGL}_{n_1}(k) \times \text{PGL}_{n_2}(k)$, is precisely the set

$$\{(\gamma^{(1)}, \gamma^{(2)}) \in \Gamma^{(1)} \times \Gamma^{(2)} \mid p^{(1)}(\gamma^{(1)}) = p^{(2)}(\gamma^{(2)})\} = \Gamma^{(1)} \times_G \Gamma^{(2)}.$$

We will write $N^{(1)}$ for $\ker p^{(1)}$, and $N^{(2)}$ for $\ker p^{(2)}$. We will write p for the obvious homomorphism $\Gamma \rightarrow G$ sending $(\gamma^{(1)}, \gamma^{(2)})$ to the common value of $p^{(1)}(\gamma^{(1)})$ and $p^{(2)}(\gamma^{(2)})$.

We are given that $q_1^{-1}(\Gamma^{(1)})$ and $q_2^{-1}(\Gamma^{(2)})$ are sturdy, from which we deduce the existence of sequences of subgroups

$$\Gamma_1^{(1)} \triangleleft \Gamma_2^{(1)} \triangleleft \cdots \triangleleft \Gamma_r^{(1)} = \Gamma^{(1)}, \quad \Gamma_1^{(2)} \triangleleft \Gamma_2^{(2)} \triangleleft \cdots \triangleleft \Gamma_s^{(2)} = \Gamma^{(2)}$$

such that for each sequence each group is normal in the next, with $q_i^{-1}(\Gamma_1^{(i)})$ very sturdy and $(\Gamma_{j+1}^{(i)} : \Gamma_j^{(i)})$ coprime to l for each i, j . In particular $\Gamma_1^{(1)}$ and $\Gamma_2^{(1)}$ are each abstractly isomorphic to a product of finite simple groups, each of which is a simple group $\mathcal{D}(\mathcal{H}(\mathbb{F}_q))$ of Lie type with $q|l$. (See Lemma 4.1.3.) For brevity in the remainder of the proof, we will refer to such a product as a l -Lie-ish group. We remark that any quotient of an l -Lie-ish group is again an l -Lie-ish group, since any normal subgroup of a product of simple perfect groups is a product of some subset of the factors.

From these, we get sequences of subgroups

$$G_1^{(1)} \triangleleft G_2^{(1)} \triangleleft \cdots \triangleleft G_r^{(1)} = G, \quad G_1^{(2)} \triangleleft G_2^{(2)} \triangleleft \cdots \triangleleft G_s^{(2)} = G$$

of G (such that again for each sequence each group is normal in the next) by putting $G_j^{(i)} = p^{(i)}(\Gamma_j^{(i)}) \cong N^{(i)}\Gamma_j^{(i)}/N^{(i)}$. Since $G_1^{(1)}$ is a quotient of $\Gamma_1^{(1)}$, which is l -Lie-ish, $G_1^{(1)}$ is also l -Lie-ish; and similarly for $G_1^{(2)}$.

Claim. *We have that $G_1^{(1)} \cap G_1^{(2)} = G_1^{(1)}$, so $G_1^{(1)} \subset G_1^{(2)}$.*

Proof. We shall show, by downward induction on j from s to 1, that $G_1^{(1)} \cap G_j^{(2)} = G_1^{(1)}$. The base case is trivial. Assuming that $G_1^{(1)} \cap G_j^{(2)} = G_1^{(1)}$, we want to understand $G_1^{(1)} \cap G_{j-1}^{(2)}$. We know $G_1^{(1)} \cap G_{j-1}^{(2)} \triangleleft G_1^{(1)} \cap G_j^{(2)} = G_1^{(1)}$. If the inclusion were proper, then $G_1^{(1)} / G_1^{(1)} \cap G_{j-1}^{(2)}$ would be a nontrivial quotient of $G_1^{(1)}$, and hence (since $G_1^{(1)}$ is l -Lie-ish) would be l -Lie-ish itself, so would have order divisible by l . But this contradicts the fact that $G_{j-1}^{(2)} : G_j^{(2)}$ and hence $G_1^{(1)} \cap G_{j-1}^{(2)} : G_1^{(1)} \cap G_j^{(2)}$ are coprime to l . \square

The same argument gives that $G_1^{(2)} \subset G_1^{(1)}$, whence we see that $G_1^{(1)} = G_1^{(2)}$. We'll adopt G_1 as a shorter name for the common value.

For each $G_j^{(1)}$, we can construct a group $\Gamma_j = p^{-1}(G_j^{(1)}) < \Gamma$; we see that each Γ_j is a normal subgroup of the next with prime-to- l index. Thus it will suffice for us to show $q^{-1}(\Gamma_1)$ is sturdy to deduce that $q^{-1}(\Gamma)$ is sturdy. We put $\Gamma' = \Gamma_1$. Note that

$$\begin{aligned} \Gamma' &= \{(\gamma^{(1)}, \gamma^{(2)}) \in \Gamma_1^{(1)} N^{(1)} \times \Gamma_1^{(2)} N^{(2)} \mid p^{(1)}(\gamma^{(1)}) = p^{(2)}(\gamma^{(2)})\} \\ &\cong \Gamma_1^{(1)} N^{(1)} \times_{G_1} \Gamma_1^{(2)} N^{(2)} \end{aligned}$$

For each j , $1 \leq j \leq r$, we can form a subgroup $N^{(1)} \Gamma_1^{(1)} \cap \Gamma_j^{(1)}$ of $N^{(1)} \Gamma_1^{(1)}$; we see that each is normal in the next, the successive quotients have order prime to l , and for each of these groups, $p^{(1)}(N^{(1)} \Gamma_1^{(1)} \cap \Gamma_j^{(1)}) = G_1$. We then form a subgroup Γ'_j of Γ' , by putting

$$\begin{aligned} \Gamma'_j &= \{(\gamma^{(1)}, \gamma^{(2)}) \in (\Gamma_1^{(1)} N^{(1)} \cap \Gamma_j^{(1)}) \times \Gamma_1^{(2)} N^{(2)} \mid p^{(1)}(\gamma^{(1)}) = p^{(2)}(\gamma^{(2)})\} \\ &\cong ((\Gamma_1^{(1)} N^{(1)}) \cap \Gamma_j^{(1)}) \times_{G_1} \Gamma_1^{(2)} N^{(2)} \end{aligned}$$

We see that each Γ'_j is normal in the next, and the successive quotients have order prime to l ; thus it will suffice for us to show $q^{-1}(\Gamma'_1)$ is sturdy to deduce that $q^{-1}(\Gamma')$ (and hence $q^{-1}(\Gamma)$) is sturdy. We put $\Gamma'' = \Gamma'_1$, so

$$\begin{aligned} \Gamma'' &= \{(\gamma^{(1)}, \gamma^{(2)}) \in \Gamma_1^{(1)} \times \Gamma_1^{(2)} N^{(2)} \mid p^{(1)}(\gamma^{(1)}) = p^{(2)}(\gamma^{(2)})\} \\ &\cong \Gamma_j^{(1)} \times_{G_1} \Gamma_1^{(2)} N^{(2)} \end{aligned}$$

Similarly for each j , $1 \leq j \leq s$, we can form a subgroup $N^{(2)} \Gamma_1^{(2)} \cap \Gamma_j^{(2)}$ of $N^{(2)} \Gamma_1^{(2)}$; we see that each is normal in the next, the successive quotients have order prime to l , and for each of these groups, $p^{(2)}(N^{(2)} \Gamma_1^{(2)} \cap \Gamma_j^{(2)}) = G_1$. We then form a subgroup Γ''_j of Γ'' , by putting

$$\begin{aligned} \Gamma''_j &= \{(\gamma^{(1)}, \gamma^{(2)}) \in \Gamma_1^{(1)} \times (\Gamma_1^{(2)} N^{(2)} \cap \Gamma_j^{(2)}) \mid p^{(1)}(\gamma^{(1)}) = p^{(2)}(\gamma^{(2)})\} \\ &\cong \Gamma_1^{(1)} \times_{G_1} ((\Gamma_1^{(2)} N^{(2)}) \cap \Gamma_j^{(2)}) \end{aligned}$$

We see that each Γ''_j is normal in the next, and the successive quotients have order prime to l ; thus it will suffice for us to show $q^{-1}(\Gamma''_1)$ is sturdy to deduce that $q^{-1}(\Gamma'')$ (and hence $q^{-1}(\Gamma)$) is sturdy. We put $\Gamma''' = \Gamma''_1$; then:

$$\begin{aligned} \Gamma''' &= \{(\gamma^{(1)}, \gamma^{(2)}) \in \Gamma_1^{(1)} \times \Gamma_1^{(2)} \mid p^{(1)}(\gamma^{(1)}) = p^{(2)}(\gamma^{(2)})\} \\ &\cong \Gamma_1^{(1)} \times_{G_1} \Gamma_1^{(2)} \end{aligned}$$

We know that the $\Gamma_1^{(i)}$ are l -Lie-ish groups, so normal subgroup is the product of a subset of the factors, and thus, writing $N_1^{(i)}$ for $\Gamma_1^{(i)} \cap N^{(i)}$, so $\Gamma_1^{(1)}/N^{(1)} \cong \Gamma_1^{(2)}/N^{(2)} \cong G_1$, we see that this isomorphism in fact identifies a subset of the factors of $\Gamma_1^{(1)}$ with a subset of the factors of $\Gamma_2^{(1)}$.

Thus (reordering the factors if necessary), we can write

$$\begin{aligned}\Gamma_1^{(1)} &\cong \prod_{i=1}^t \pi_i(\mathcal{G}_i^{sc}(k_i)) \times \prod_{i=t+1}^{m^{(1)}} \pi_i^{(1)}(\mathcal{G}_i^{(1),sc}(k_i)) \\ \Gamma_1^{(2)} &\cong \prod_{i=1}^t \pi_i(\mathcal{G}_i^{sc}(k_i)) \times \prod_{i=t+1}^{m^{(2)}} \pi_i^{(2)}(\mathcal{G}_i^{(2),sc}(k_i))\end{aligned}$$

(So the map to G_1 is, in either case, projection onto the first t factors.)

Now $\Gamma_1^{(1)}$ (as a subgroup, rather than up to isomorphism) is the image of

$$\prod_{i=1}^t \pi_i(\mathcal{G}_i^{sc}(k_i)) \times \prod_{i=t+1}^{m^{(1)}} \pi_i^{(1)}(\mathcal{G}_i^{(1),sc}(k_i)) \xrightarrow{\prod_i r_i^{(1)}} \prod_i \text{PGL}_{n_i^{(1)}}(k) \hookrightarrow \text{PGL}_{n_1}(k)$$

for some projective representations $r_i^{(1)}$, $i = 1, \dots, m^{(1)}$, of dimension $n_i^{(1)}$; and similarly for $\Gamma_1^{(2)}$ (with projective representations $r_i^{(2)}$ of dimension $n_i^{(2)}$). Thus for each i , $i \leq t$, $r_i^{(1)}$ and $r_i^{(2)}$ are projective representations of a common group $\pi_i(\mathcal{G}_i^{sc}(k_i))$, and we can form their tensor product, r_i' say.

Then we see Γ''' is (conjugate to) the image of

$$\begin{aligned}&\prod_{i=1}^t \pi_i(\mathcal{G}_i^{sc}(k_i)) \times \prod_{i=t+1}^{m^{(1)}} \pi_i^{(1)}(\mathcal{G}_i^{(1),sc}(k_i)) \times \prod_{i=t+1}^{m^{(2)}} \pi_i^{(2)}(\mathcal{G}_i^{(2),sc}(k_i)) \\ &\quad \downarrow \prod_{i=1}^t r_i' \times \prod_{i=t+1}^{m^{(1)}} r_i^{(1)} \times \prod_{i=t+1}^{m^{(2)}} r_i^{(2)} \\ &\prod_{i=1}^t \text{PGL}_{n_i^{(1)} n_i^{(2)}}(k) \times \prod_{i=t+1}^{m^{(1)}} \text{PGL}_{n_i^{(1)}}(k) \times \prod_{i=t+1}^{m^{(2)}} \text{PGL}_{n_i^{(2)}}(k) \\ &\quad \downarrow \\ &\text{PGL}_n(k).\end{aligned}$$

This demonstrates that Γ''' is very sturdy and hence sturdy, so Γ is sturdy. \square

Using a few of the same ideas, we can also prove:

Lemma 4.2.2. *Suppose l is a prime, n is a positive integer, k/\mathbb{F}_l is a finite extension, Γ is a sturdy subgroup of $\text{GL}_n(k)$, and N is a normal subgroup of index prime to l . Then N is also sturdy.*

Proof. By assumption, we have a sequence $\Gamma_1 \triangleleft \Gamma_2 \triangleleft \dots \triangleleft \Gamma_r = \Gamma$ of subgroups, each normal in the next with index prime to l , and with Γ_1 very sturdy. Then we have a sequence $\Gamma_1 \cap N \triangleleft \Gamma_2 \cap N \triangleleft \dots \triangleleft \Gamma_r \cap N = N$ of subgroups of N , each normal in the next with index prime to l ; so if we show $\Gamma_1 \cap N$ very sturdy we will be done. $\Gamma_1 \cap N : \Gamma_1$ is prime to l , whereas Γ_1 is l -Lie-ish (as discussed in the proof above), so every proper quotient is l -Lie-ish and hence has order dividing l . So $\Gamma_1 \cap N = \Gamma_1$ and the lemma follows. \square

4.3. Very regular elements. In the next subsection, we will prove a result showing that sturdy subgroups are big for l large. But before we turn to this task, we first need a lemma allowing the construction of ‘very regular elements’, as in [SW10, §4] and [Whi10, §3]. Our arguments will have to be a little more complicated, however, since we will have a collection of different algebraic groups (defined over different fields), and we need to elements in each which are ‘very regular in combination’.

The arguments in this section are rather technical but are elementary and involve little of enduring interest. We might advise the reader to skip them at a first reading, studying only the statement of Lemma 4.3.5 before moving on to the next subsection. (We remark that had we been willing to introduce a bound on the degree $[k : \mathbb{F}_l]$ in the conditions on our main theorems, and let the bounds on l in the main theorems depend on the bound on this degree (which is probably harmless in most applications), we could have given radically simpler proofs of our results here, essentially appealing to results of [SW10] applied to appropriate restrictions of scalars.)

We will establish a succession of stronger and stronger lemmas building up to the main result of the subsection.

Lemma 4.3.1. *Suppose Ω, Ξ, N are positive integers. There is an integer L with the following property.*

Suppose that d is a positive integer and $\vec{\mu} = (\mu_0, \dots, \mu_{d-1}) \in (\mathbb{Z} \cap [-N, N])^d$, so each μ_i is an integer between $-N$ and N . Suppose also that at most Ξ of the μ_i are nonzero, and that $q = p^k$ is a prime power, with $p^k | d$, such that

$$\sum_{i=0, \dots, d-1} \zeta_d^{p^k i} \mu_i \neq 0$$

Now suppose finally that $l > L$ is a prime number, and let $h = (\sum_j \mu_j l^j)$. Then $\#\{ht | t \in \mathbb{Z}/(l^d - 1)\mathbb{Z}\} > (\sqrt[d]{\Omega})^{d/\log \log d}$.

Proof. We begin by choosing the number L . We first recall that there is a constant K such that $\phi(n) > Kn/\log \log n$, where ϕ is Euler’s ϕ function—see [HW54, Theorem 328]. We then take L to be large enough that $L > 2\Xi \sqrt[d]{\Omega}N$.

We now move on to the proof proper. We first consider the linear map $\psi : \mathbb{Q}^d \rightarrow \mathbb{Q}^d$ given by ‘convolution by μ ’—that is, the map

$$(t_0, \dots, t_{d-1}) \mapsto \left(\sum_{i=0}^d t_i \mu_{-i}, \sum_{i=0}^d t_i \mu_{1-i}, \dots, \sum_{i=0}^d t_i \mu_{d-1-i} \right)$$

—and we claim it has image which is at least $\phi(d/p^k)$ dimensional. To see this, we note that as ‘convolution is Fourier dual to pointwise multiplication’, it suffices to show that, if we write $(\hat{\mu}_0, \dots, \hat{\mu}_{d-1})$ for the Fourier transform of μ , so $\hat{\mu}_i = \sum_j \zeta_d^{ij} \mu_j$, then the linear map $\mathbb{Q}^d \rightarrow \mathbb{Q}^d$ defined by

$$(\hat{t}_0, \dots, \hat{t}_{d-1}) \mapsto (\hat{t}_0 \hat{\mu}_0, \hat{t}_1 \hat{\mu}_1, \dots, \hat{t}_{d-1} \hat{\mu}_{d-1})$$

has image which is at least $\phi(d/p^k)$ dimensional. In other words, we must show that at least $\phi(d/p^k)$ of the $\hat{\mu}_j$ are nonzero. But, by hypothesis, we know that $\sum_{i=0, \dots, d-1} \zeta_d^{p^k i} \mu_i \neq 0$; and applying elements of $\text{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})$ we see that $\sum_{i=0}^{d-1} \zeta_d^{p^k \eta i} \mu_i \neq 0$ for all η coprime to d . This tells us that $\hat{\mu}_j$ is nonzero whenever $p^k | j$ and j/p^k is coprime to d/p^k . There are $\phi(d/p^k)$ such numbers. This proves the claim in the first sentence of the paragraph.

We note that we can consider ψ also as a map $\mathbb{Z}^d \rightarrow \mathbb{Z}^d$. Then the result of the previous paragraph tells us that the cardinality of the image $\psi([0, \sqrt[k]{\Omega}] \cap \mathbb{Z})^d$ satisfies:

$$\#\psi([0, \sqrt[k]{\Omega}] \cap \mathbb{Z})^d > (\sqrt[k]{\Omega})^{\phi(d/p^k)}$$

and we then deduce moreover that this

$$> (\sqrt[k]{\Omega})^{\frac{Kd}{p^k \log(d/p^k)}} > (\sqrt[k]{\Omega})^{\frac{Kd}{p^k \log(d)}} = (\sqrt[q]{\Omega})^{\frac{d}{\log(d)}}.$$

On the other hand, examining the image $\psi([0, \sqrt[k]{\Omega}] \cap \mathbb{Z})^d$, we see that for each $(\nu_0, \dots, \nu_{d-1}) \in \psi([0, \sqrt[k]{\Omega}] \cap \mathbb{Z})^d$, each component ν_i is a sum $\sum_{j=0}^d t_j \mu_{i-j}$ with at most Ξ nonzero terms (since at most Ξ of the μ_i are nonzero), each term of which is a product of an integer $< \sqrt[k]{\Omega}$ and an integer between $-N$ and N . Thus

$$\nu_i \in [-\Xi N \sqrt[k]{\Omega}, \Xi N \sqrt[k]{\Omega}]$$

for each i and ν . Thus $\psi([0, \sqrt[k]{\Omega}] \cap \mathbb{Z})^d$ lies in the set

$$S := ([-\Xi N \sqrt[k]{\Omega}, \Xi N \sqrt[k]{\Omega}] \cap \mathbb{Z})^d.$$

But we observe that the map $\alpha : S \rightarrow \mathbb{Z}/(l^d - 1)\mathbb{Z}$ mapping $(\nu_0, \dots, \nu_{d-1}) \mapsto \sum \nu_i l^i$ is injective. (This depends on the fact that $l > L > 2\Xi \sqrt[k]{\Omega} N$.) Thus $\#\alpha(\psi([0, \sqrt[k]{\Omega}] \cap \mathbb{Z})^d) > (\sqrt[q]{\Omega})^{d/\log(d)}$.

On the other hand, for $(t_0, \dots, t_{d-1}) \in \mathbb{Z}^d$, we have that

$$\begin{aligned} \alpha(\psi(t_0, \dots, t_{d-1})) &= \sum_i l^i \sum_j t_j \mu_{i-j} \pmod{l^d - 1} \\ &= \sum_j \sum_m t_j l^{j+m} \mu_m \end{aligned}$$

(putting $i = j + m$ and changing the order of summation; notice that not only do we take the subscripts mod d , but we can also take the power of l mod d , since we work overall mod $l^d - 1$)

$$= h \sum_j t_j l^j \in \{ht | t \in \mathbb{Z}/(l^d - 1)\mathbb{Z}\}$$

whence $\#\{ht | t \in \mathbb{Z}/(l^d - 1)\mathbb{Z}\} > \#\alpha(\psi([0, \sqrt[k]{\Omega}] \cap \mathbb{Z})^d) > (\sqrt[q]{\Omega})^{d/\log(d)}$, as required. \square

Lemma 4.3.2. *Suppose that Ξ is a positive integer. There is an integer n with the following property.*

Suppose that d is a positive integer, and $\vec{\mu} = (\mu_0, \dots, \mu_{d-1}) \in \mathbb{Z}^d$. Suppose that both of the following conditions hold:

- *We have $\#\{i | \mu_i \neq 0\} < \Xi$.*
- *Whenever p^k is a prime power, with $p^k < n$ and $p^k | d$, we have that*

$$\sum_{i=0, \dots, d-1} \zeta_d^{p^k i} \mu_i = 0$$

where ζ_d denotes a primitive d th root of 1.

Then $\vec{\mu}$ is identically 0.

Proof. We take $n = \Xi + 1$. Let us begin by establishing some notation. Let V_1 denote the vector space of \mathbb{Q} -valued functions on the group $\mathbb{Z}/d\mathbb{Z}$. We think of $\vec{\mu}$ as an element of V_1 . For each positive integer q dividing d , let V_d denote the subspace of functions f such that $f(a + (d/q)) = f(a)$ for all $a \in \mathbb{Z}/d\mathbb{Z}$. For each $d'|d$, let $W_1^{(d')}$ denote the vector space of \mathbb{Q} -valued functions on $\mathbb{Z}/d'\mathbb{Z}$, for each prime power $q|d'$, let $W_q^{(d')}$ denote the set $\{f \in W^{(d')} \mid f(a + (d'/q)) = f(a) \text{ for all } a \in \mathbb{Z}/d'\mathbb{Z}\}$. Thus $W_q^{(d')}$ is an alias for V_q . Finally, we note that we have a map $\pi_{d'} : V_1 \rightarrow W_1^{(d')}$ mapping a function $f : \mathbb{Z}/d\mathbb{Z} \rightarrow \mathbb{Q}$ to the function g sending $a \in \mathbb{Z}/d'\mathbb{Z}$ to the sum of f over elements $b \in \mathbb{Z}/d\mathbb{Z}$ which give a when reduced mod d' . There is a similar map $\pi_{d',d''} : W_1^{(d')} \rightarrow W_1^{(d'')}$ for all $d''|d'$.

Claim. Suppose that $\sum_{i=0}^{d-1} \zeta_d^i \mu_i = 0$. Then $\vec{\mu} \in \sum_{p \mid d} V_p$, where the sigma denotes the internal sum of subspaces of V_1 .

Proof. We consider $\mathbb{Q}(\zeta_d)$ as a \mathbb{Q} vector space (which is $\phi(d)$ dimensional, by the standard theory of cyclotomic extensions), and consider the linear map of \mathbb{Q} vector spaces $\alpha : V_1 \rightarrow \mathbb{Q}(\zeta_d)$ mapping $f \mapsto \sum_{i=0}^{d-1} f(i) \zeta_d^i$, which is clearly surjective. Hence $\ker \alpha$ is $d - \phi(d)$ dimensional. On the other hand, for each prime p dividing d , we readily see that $V_p \subset \ker \alpha$, so that $\sum_{p \in P} V_p \subset \ker \alpha$, where P denotes the set of primes dividing d . On the other hand,

$$\begin{aligned} \dim \sum_{p|d, p \text{ prime}} V_p &= \sum_{S \subset P, S \neq \emptyset} (-1)^{\#S} \dim \left(\bigcap_{p \in S} V_p \right) \\ &= \sum_{S \subset P, S \neq \emptyset} (-1)^{\#S} \frac{d}{\prod_{p \in S} p} = d - \phi(d). \end{aligned}$$

It follows that $\ker \alpha = \dim \sum_{p|d, p \text{ prime}} V_p$, and hence, as we assume $\vec{\mu} \in \ker \alpha$, our conclusion follows. \square

The same ideas readily extend to prove:

Claim. Suppose that $q|d$ is a prime power and $\sum_{i=0}^{d-1} \zeta_d^{qi} \mu_i = 0$. Then

$$\vec{\mu} \in \pi_{d/q}^{-1} \left(\sum_{p \mid d/q} W_p^{(d/q)} \right).$$

Now, suppose that q_1, \dots, q_r are all prime powers dividing d , and write q'_1 for the next-highest power of the prime dividing q_1 after q_1 (so if $q_1 = p^k$, $q'_1 = p^{k+1}$). Then we see that:

$$\begin{aligned} (4.3.1) \quad \left(\sum_{i=1, \dots, r} V_{d/q_i} \right) \cap \left(\pi_{d/q_1} \left(\sum_{p \mid d/q_1} W_p^{(d/q_1)} \right) \right) \\ = \begin{cases} V_{d/q'_1} + \sum_{i=2, \dots, r} V_{d/q_i} & (\text{if } q'_1 | d) \\ \sum_{i=2, \dots, r} V_{d/q_i} & (\text{if } q'_1 \nmid d) \end{cases} \end{aligned}$$

We are now in a position to proceed to the proof proper. Suppose that $\vec{\mu}$ satisfies the two bullet points in the statement of the proposition. Let p_1, \dots, p_r be the primes dividing d . We know from the first claim above that $\vec{\mu} \in V_{p_1} + \dots + V_{p_r}$. We

will think of this as saying $\vec{\mu} \in V_{p_1^{a_1}} + \dots + V_{p_r^{a_r}}$, where each a_i stands for 1 for the time being. Our aim will be to gradually increase the numbers a_i and/or reduce the number of terms in the sum $V_{p_1^{a_1}} + \dots + V_{p_s^{a_s}}$, always maintaining the fact that

$$(4.3.2) \quad \vec{\mu} \in V_{p_1^{a_1}} + \dots + V_{p_s^{a_s}}$$

for some $s \leq r$. (As we go along, we will allow ourselves to reorder the primes p_1, \dots, p_r .)

Now, let us suppose that at least one of the $p_i^{a_i}$ is $< n$. We know that

$$\vec{\mu} \in \pi_{d/p_i^{a_i}}^{-1} \left(\sum_{p \text{ a prime}, p|(d/p_i^{a_i})} W_p^{(d/p_i^{a_i})} \right)$$

and hence, using equation 4.3.1 and the ongoing assumption in equation 4.3.2, we can either replace the term $V_{p_i^{a_i}}$ in the equation 4.3.2 with $V_{p_i^{a_i+1}}$ or suppress the term entirely (in which case we can reorder the p_j s such that $p_i = p_s$, and then replace s with $s - 1$, so maintaining the form of equation 4.3.2).

After repeating this procedure a finite number of times (which can be bounded in terms of the number of distinct prime powers dividing d), we see that $\vec{\mu} \in V_{p_1^{a_1}} + \dots + V_{p_r^{a_r}}$ where each $p_i^{a_i}$ is $\geq n$. We let q_1, \dots, q_s be the prime powers $p_1^{a_1}, \dots, p_s^{a_s}$.

We know then that $\vec{\mu}$ can be written as a sum $v_1 + \dots + v_s$ where each $v_i \in V_{q_i}$. On the other hand, this expression is of course not unique, since the map $\oplus_i V_{q_i} \rightarrow \sum_i V_{q_i}$ is not injective. We place an ordering on tuples $(v_1, \dots, v_s) \in \oplus_i V_{q_i}$ with $\sum_i v_i = \vec{\mu}$ as follows:

- *First*, we order tuples by the number of initial zeros in the tuple. That is, we order according to the maximum t such that v_1, \dots, v_{t-1} are all 0. (The more zeros, the ‘larger’ we consider the tuple to be.)
- *Second*, amongst tuples with the same number of initial zeros, we order as follows. Let v_t be the first nonzero element in the tuple. v_t is an element of V_{q_t} , and hence a function on $\mathbb{Z}/d\mathbb{Z}$. We order according to the number of times this function takes a nonzero value, with fewer nonzero values counting as making the tuple ‘larger’.

It is easy to see that we can find a maximal tuple (v_1, \dots, v_s) in this collection. It is possible that this maximal tuple is in fact $(0, \dots, 0)$, whence $\vec{\mu} = 0$ and we are done. So let us suppose that this is not the case, so there is some t with v_1, \dots, v_{t-1} all 0 but v_t nonzero.

We write U for $\sum_{i>t} V_{q_i q_t}$, a subset of V_{q_t} . We see that we cannot find a $u \in U$ with $v_t - u$ (an element of V_{q_t} , and hence a function on $\mathbb{Z}/d\mathbb{Z}$) taking fewer nonzero values than v_t does, else we could write $u = \sum_{i>t} u_i$ where $u_i \in V_{q_i q_t} \subset V_{q_i} \cap V_{q_t}$, and replace the tuple $(0, \dots, 0, v_t, v_{t+1}, \dots, v_s)$ with $(0, \dots, 0, v_t - u, v_{t+1} + u_{t+1}, \dots, v_s + u_s)$ which is larger according to the ordering above, contradicting the maximality of (v_1, \dots, v_s) .

Now, for each $i \in \mathbb{Z}/(d/q_t)\mathbb{Z}$ we write S_i for the set of elements of $\mathbb{Z}/d\mathbb{Z}$ congruent to $i \bmod d/q_t$. We see that the number of nonzero values that $\vec{\mu}$ takes on the subset S_t must be at least the number of nonzero values v_t takes on what subset, otherwise we could find a u as in the previous paragraph. But v_t is nonzero and periodic with period d/q_t . Thus v_t is nonzero for at least $q_t > n > \Xi$ possible inputs, so $\vec{\mu}$ is too, contradicting the first bullet point in the statement of the lemma. \square

Lemma 4.3.3. *Suppose Ξ , Φ , and N are positive integers, and $\epsilon > 0$ is a real number. Then we can find an integer $L(N, \Phi, \Xi, \epsilon)$ with the following property.*

Suppose that $l > L(N, \Phi, \Xi)$ is a prime, $\nu < \Phi$ is an integer, and $d \in \mathbb{Z}_{\geq 1}$. Suppose that for each $i = 1, \dots, \nu$ we are given a $d_i \in \mathbb{Z}_{\geq 1}$ with $d_i | d$. Write \tilde{T} for the finite (additive) group

$$\tilde{T} = \left\{ (t_1, \dots, t_\nu) \in \prod_{i=1, \dots, \nu} \mathbb{Z}/(l^{d_i} - 1)\mathbb{Z} \mid l^{d_i} t_i = t_i \text{ for all } i \right\}.$$

Write S for the set

$$S = \prod_{i=1, \dots, \nu} \prod_{j=0, \dots, d_i} (\mathbb{Z} \cap [-N, N]),$$

so we write a typical element of S as $\vec{\mu} = ((\mu_{1,0}, \dots, \mu_{1,d_1}), \dots, (\mu_{\nu,0}, \dots, \mu_{\nu,d_\nu}))$.

Write S_Ξ for the subset of S consisting of elements μ for which $\#\{(i, j) | \mu_{i,j} \neq 0\} < \Xi$.

For $\vec{\mu} \in S$ and $t \in \tilde{T}$, we define

$$\vec{\mu}(t) = \sum_{i=1, \dots, \nu} \sum_{j=0, \dots, d_i} l^j \mu_{i,j} t_i \quad (\text{so } \vec{\mu}(t) \in \mathbb{Z}/(l^d - 1)\mathbb{Z}).$$

Then we have

$$\#\{(\vec{\mu}, t) \in S_\Xi \times \tilde{T} | \vec{\mu}(t) = 0\} \leq \epsilon \#\tilde{T}.$$

Proof. Step 1: selecting L . We apply Lemma 4.3.2 with Ω and Ξ as in the present context, deducing the existence of an integer n with the property described there. We write Q for the set of tuples (q_1, \dots, q_ν) where each q_i is either a prime power $< n$, or the symbol ∞ ; and we note that this set is finite. We set $\delta = \epsilon/\#Q$.

It then is a trivial exercise in analysis that we can choose Ω large enough that

$$\frac{(\Xi^\Phi)^x (2N)^\Xi}{(\sqrt[n]{\Omega})^{x/\log \log x}} < \delta$$

for all $x \geq 1$. We then apply Lemma 4.3.1 with this choice of Ω , and with Ξ and N as in the present context; then lemma furnishes a choice of L with a certain property (as described there). This is the L we will use.

Now we suppose that l, ν, d etc. are as in the statement of the lemma.

Step 2: introducing the sets $S(\vec{q})$ for each $\vec{q} = (q_1, \dots, q_\nu) \in Q$, and showing $\bigcup_{\vec{q} \in Q} S(\vec{q}) \supset S_\Xi$. For each $\vec{q} \in Q$ with the property that, for each i , either $q_i = \infty$, or $q_i | d_i$, we write $S(\vec{q})$ for the subset of S_Ξ consisting of elements $\vec{\mu}$ for which, for those $i \in \{1, \dots, \nu\}$ where q_i is a prime power, we have that:

$$\sum_{j=0, \dots, d_i-1} \zeta_d^{q_i j} \mu_{i,j} \neq 0$$

while for those i where q_i is ∞ , we have that $\mu_{i,j} = 0$. For $\vec{q} \in Q$ with $q_i \nmid d_i$ for some i , we set $S(\vec{q}) = \emptyset$.

By application of Lemma 4.3.2, we see that for each fixed $\vec{\mu} \in S_\Xi$, and each fixed i , there either is some prime power $q_i < n$ dividing d_i such that the sum in the displayed equation is nonzero, or else we have that $\mu_{i,j}$ is zero for all j (this is since $\#\{j | \mu_{i,j} \neq 0\} < \Xi$). It follows that for each fixed $\vec{\mu} \in S_\Xi$, we can find a tuple $\vec{q} \in Q$ such that the sum in the displayed equation is nonzero for every i , and hence it follows that $\bigcup_{\vec{q} \in Q} S(\vec{q}) \supset S_\Xi$.

Step 3: bounding $\#\{(\vec{\mu}, t) \in S(\vec{q}) \times \tilde{T} | \vec{\mu}(t) = 0\}$ for each \vec{q} . We now fix $\vec{q} \in Q$. Our goal in this step of the proof is to show $\#\{(\vec{\mu}, t) \in S(\vec{q}) \times \tilde{T} | \vec{\mu}(t) = 0\} \leq \delta \#\tilde{T}$. If $q_i \nmid d_i$ for some i , then $S(\vec{q}) = \emptyset$ and there is nothing to prove, so we may assume $q_i | d_i$ or $q_i = \infty$ for all i . Moreover, for those i for which $q_i = \infty$, we can simply imagine that the corresponding $\mu_{i,j}$ (and t_i) no longer exist (i.e. ν is reduced) and then suppress those q_i . So we can assume that $q_i | d_i$ for all i .

Let $d_{\max} = \max_i d_i$. To give an element of $S(\vec{q})$, it suffices to choose Ξ pairs (i, j) (this chooses Ξ of the $\mu_{i,j}$ to be potentially nonzero), then for each of these Ξ of the $\mu_{i,j}$ we must choose their value. The first choice may be made in at most $\Xi \sum_i d_i$ ways, and the second in fewer than $(2N)^\Xi$ ways. Thus

$$\#S(\vec{q}) \leq \Xi \sum_i d_i (2N)^\Xi < \Xi^{\Phi d_{\max}} (2N)^\Xi.$$

Next let us fix some $\vec{\mu} \in S(\vec{q})$. Let i be some index such that $d_i = d_{\max}$. We may think of $\vec{\mu}$ as determining a homomorphism $t \mapsto \vec{\mu}(t)$ mapping $\tilde{T} \rightarrow \mathbb{Z}/(l^d - 1)\mathbb{Z}$, and we may think of $\vec{\mu}_i = (\mu_{i,0}, \dots, \mu_{i,d_i-1})$ as determining a homomorphism $\mathbb{Z}/(l^{d_{\max}} - 1)\mathbb{Z} \rightarrow \mathbb{Z}/(l^{d_{\max}} - 1)\mathbb{Z}$, where $t \mapsto \sum_j l^j \mu_{i,j} t$. We have that

$$\begin{aligned} \#\{t \in \tilde{T} | \vec{\mu}(t) = 0\} &= \#\ker \vec{\mu} = \frac{\#\tilde{T}}{(\tilde{T} : \ker \vec{\mu})} = \frac{\#\tilde{T}}{\#\vec{\mu}(\tilde{T})} \\ &\leq \frac{\#\tilde{T}}{\#\vec{\mu}_i(\mathbb{Z}/(l^{d_{\max}} - 1)\mathbb{Z})} < \frac{\#\tilde{T}}{(\sqrt[q_i]{\Omega})^{d_{\max}/\log \log d_{\max}}} \\ &< \frac{\#\tilde{T}}{(\sqrt[\nu]{\Omega})^{d_{\max}/\log \log d_{\max}}} \end{aligned}$$

(where the penultimate inequality relies on our choice of $l > L$, where L was the constant from Lemma 4.3.1).

Combining the bounds on $\#S(\vec{q})$ and, for each $\vec{\mu} \in S(\vec{q})$, on $\#\{t \in \tilde{T} | \vec{\mu}(t) = 0\}$, we see that

$$\#\{(\vec{\mu}, t) \in S(\vec{q}) \times \tilde{T} | \vec{\mu}(t) = 0\} \leq \frac{\#\tilde{T}}{(\sqrt[\nu]{\Omega})^{d_{\max}/\log \log d_{\max}}} \Xi^{\Phi d_{\max}} (2N)^\Xi < \delta \#\tilde{T}$$

completing this step of the proof.

Step 4: concluding the argument. We may write, using step 2,

$$\{(\vec{\mu}, t) \in S_\Xi \times \tilde{T} | \vec{\mu}(t) = 0\} \subset \bigcup_{\vec{q} \in Q} \{(\vec{\mu}, t) \in S(\vec{q}) \times \tilde{T} | \vec{\mu}(t) = 0\}$$

and hence

$$\begin{aligned} \#\{(\vec{\mu}, t) \in S_\Xi \times \tilde{T} | \vec{\mu}(t) = 0\} &\leq \sum_{\vec{q} \in Q} \#\{(\vec{\mu}, t) \in S(\vec{q}) \times \tilde{T} | \vec{\mu}(t) = 0\} \\ &\leq \sum_{\vec{q} \in Q} \delta \#\tilde{T} = \#Q \delta \#\tilde{T} = \epsilon \#\tilde{T}. \end{aligned}$$

This is as required. \square

Lemma 4.3.4. *Suppose Ξ , Φ , and N are positive integers, and $\epsilon > 0$ is a real number. Then we can find an integer $L(N, \Phi, \Xi)$ with the following property.*

Suppose that $l > L(N, \Phi, \Xi)$ is a prime, $\nu < \Phi$ is an integer, and that k/\mathbb{F}_l is a finite extension. Suppose that for each $i = 1, \dots, \nu$ we are given a field k_i , with

$\mathbb{F}_l \subset k_i \subset k$. Write $d_i = [k_i : \mathbb{F}_l]$. Write T^* for the finite group $\prod_i \mathbb{G}_m(k_i)$. Write S for the set

$$S = \prod_{i=1, \dots, \nu} \prod_{j=0, \dots, d_i} (\mathbb{Z} \cap [-N, N]).$$

Write S_Ξ for the subset of S consisting of elements μ for which $\#\{(i, j) | \mu_{i,j} \neq 0\} < \Xi$.

For $\vec{\mu} \in S$ and $t \in T^*$, we define

$$\vec{\mu}(t) = \prod_{i=1, \dots, \nu} \prod_{j=0, \dots, d_i} t_i^{l^j \mu_{i,j}} \in k^\times.$$

Then we have

$$\#\{(\vec{\mu}, t) \in S_\Xi \times T^* | \vec{\mu}(t) = 1\} \leq \epsilon \#T^*,$$

and hence, if we have taken $\epsilon < 1$, there is some $t \in T^*$ such that for no $\vec{\mu} \in S_\Xi$ do we have $\vec{\mu}(t) = 1$.

Proof. The first part of the lemma reduces to the previous one after the application of some choice of a ‘discrete log’ isomorphism, $\mathbb{G}_m(k) \xrightarrow{\sim} \mathbb{Z}/(l^{[k:\mathbb{F}_l]} - 1)\mathbb{Z}$. The second is immediate by considering cardinalities. \square

Lemma 4.3.5. *Suppose that n , M and Ξ' are a positive integers. There is a constant $C_2(M, n, \Xi')$ with the following property. Suppose $l > C_2(M, n)$ is a prime, and we have an integer m , fields k_i , and groups \mathcal{G}_i^{sc} exhibiting some subgroup $\Gamma < \mathrm{GL}_n(k)$ as very sturdy, as per Definition 4.1.1; suppose further that these have the additional properties described in Remark 4.1.4. Suppose for each i , \mathcal{T}/k_i is a maximal torus in \mathcal{G}_i . Then we can find elements $g_i \in \mathcal{T}_i(k_i) \subset \mathcal{G}_i(k_i)$ for each i , such that the map*

$$\begin{aligned} \alpha : \prod_{i=1, \dots, m} \{ \lambda \in X(\mathcal{T}_{i, \bar{k}}) | \|\lambda\| < C_1(n) \}^{[k_i, \mathbb{F}_l]} &\rightarrow \bar{k}^\times \\ ((\lambda_{1,1}, \dots, \lambda_{1,[k_1, \mathbb{F}_l]}), \dots, (\lambda_{m,1}, \dots, \lambda_{m,[k_m, \mathbb{F}_l]})) &\mapsto \left(\prod_i \prod_{j=0, \dots, [k_1: \mathbb{F}_l]} \lambda_{i,j} (g_i)^{l^j} \right)^M \end{aligned}$$

is injective on the subset of the domain consisting of $\vec{\lambda}$ with at most Ξ' of the $\lambda_{i,j}$ nonzero. (Here $X(\mathcal{T}_{i, \bar{k}})$ denotes the set of weights for $\mathcal{T}_{i, \bar{k}}$.)

Proof. Before we choose the constant $C_2(M, n)$, let us imagine briefly that we have an integer m , fields k_i , and groups \mathcal{G}_i^{sc} exhibiting some subgroup $\Gamma < \mathrm{GL}_n(k)$ as very sturdy in order to introduce some notation. As per Remark 4.1.4, for each i , $\mathcal{T}_i(k_i)$ can be written as $\prod_{o \in \mathcal{O}_i} \mathbb{G}_m(k_{i,o})$, where \mathcal{O}_i is the set of orbits of the simple roots in $X(\mathcal{T}_i)$ under $\mathrm{Gal}(\bar{k}/k_i)$, o stands for a particular orbit, and $k_{i,o}$ is an extension of k_i with $[k_i : k_{i,o}] = \#o$. We can associate to a tuple $(\lambda_{i,1}, \dots, \lambda_{i,[k_i, \mathbb{F}_l]})$ a collection of tuples $(\mu_{i,o,1}, \dots, \mu_{i,o,[k_i,o:\mathbb{F}_l]})$ such that, whenever

- (g_1, \dots, g_m) is a tuple of elements in $\prod_i \mathcal{T}_i(k_i)$, which corresponds to a tuple of elements

$$((t_{1,1}, \dots, t_{1,\# \mathcal{O}_1}), (t_{2,1}, \dots, t_{2,\# \mathcal{O}_2}), \dots, (t_{m,1}, \dots, t_{m,\# \mathcal{O}_m})) \in \prod_{i=1, \dots, m} \prod_{o \in \mathcal{O}_i} \mathbb{G}_m(k_{i,o})$$

under the isomorphism $\mathcal{T}_i(k_i) \cong \prod_{o \in \mathcal{O}_i} \mathbb{G}_m(k_{i,o})$,

we have that

$$\prod_{i=1,\dots,m} \prod_{j=0,\dots,[k_1:\mathbb{F}_l]} \lambda_{i,j}(g_i)^{Ml^j} = \prod_{i=1,\dots,m} \prod_{o \in \mathcal{O}_i} \prod_{j=0,\dots,[k_{i,o}:\mathbb{F}_l]} t_{i,j}^{\mu_{i,o,j} l^j}$$

Finally, we write $\nu = \sum_i \#\mathcal{O}_i$.

At this point we will proceed to choose $C_2(M, n)$, which must not depend on the integer m , fields k_i , and groups \mathcal{G}_i^{sc} . The fact that each $\lambda_{i,j}$ has $\|\lambda_{i,j}\| < C_1(n)$ allows us to find an integer N depending only on M and n such that whenever we have $\mu_{i,o,j}$ s as above, $-N < 2\mu_{i,o,j} < N$ for all i, o, j .

We can also bound in terms of n the largest possible rank of any of the groups \mathcal{G}_i , (since the group must have a faithful mod center representation of dimension $< n$). Thus, if we assume at most $2\Xi'$ of the λ are nonzero, since we can bound how many μ s are associated to each of these λ (using the bound on the rank of the \mathcal{G}_i), and because only the μ s associated to a nonzero λ may possibly be nonzero, we deduce that there is a constant Ξ depending on n and Ξ' alone such that at most Ξ of the μ s are nonzero.

Similarly, we can bound ν in terms of n alone, since we first bound m in terms of n and then use the bound on the ranks of the \mathcal{G}_i to bound the $\#\mathcal{O}_i$. Let Φ be the upper bound on ν .

We apply Lemma 4.3.4 with

- Φ , Ξ and N as in the present context, and
- $\epsilon = 0.9$.

We get a constant L , and we put $C_2(M, n, \Xi') = L$.

We then see that if we have an integer m , fields k_i , and groups \mathcal{G}_i^{sc} exhibiting some subgroup $\Gamma < \mathrm{GL}_n(k)$ as very sturdy, and we apply the property of the bound L given in Lemma 4.3.4, we have that we can find a $t \in \prod_i \prod_{o \in \mathcal{O}_i} \mathbb{G}_m(k_{i,o})$ (corresponding to an element (g_1, \dots, g_m) , in $\prod_i \mathcal{T}_i(k_i) \subset \prod_i \mathcal{G}_i(k_i)$, with the following property:

- For any $\lambda \in \prod_{i=1,\dots,m} \{\lambda \in X(\mathcal{T}_{i,\bar{k}}) \mid \|\lambda\| < C_1(n)\}^{[k_i:\mathbb{F}_l]}$ with at most $2\Xi'$ of the $\lambda_{i,j}$ nonzero (and hence with at most Ξ of the corresponding μ s nonzero) we have that

$$\prod_{i=1,\dots,m} \prod_{j=0,\dots,[k_1:\mathbb{F}_l]} \lambda_{i,j}(g_i)^{Ml^j} = \prod_{i=1,\dots,m} \prod_{o \in \mathcal{O}_i} \prod_{j=0,\dots,[k_{i,o}:\mathbb{F}_l]} t_{i,j}^{\mu_{i,o,j} l^j}$$

is never 1.

It follows that the map α given in the statement of the lemma is injective on the subset of its domain as described there, if $\vec{\lambda}, \vec{\lambda}'$ are two elements in the domain with at most Ξ' of the $\lambda_{i,j}$ and $\lambda'_{i,j}$ nonzero, then at most $2\Xi'$ of the $(\lambda/\lambda')_{i,j}$ are nonzero, and so $\alpha(\lambda/\lambda')$ cannot equal 1, by the bullet point immediately above. \square

4.4. Sturdy implies big. Our next goal is to show that sturdy subgroups are automatically big, at least for l large. Our arguments draw will very heavily on [SW10, Whi10].

Proposition 4.4.1. *Let M and n be positive integers. There is a constant $C_3(M, n)$ depending only on M and n with the following property: if l is a prime number which is larger than $C_3(M, n)$, k/\mathbb{F}_l a finite extension, and $\Gamma < \mathrm{GL}_n(k)$ a very sturdy subgroup, then Γ is M -big.*

Proof. Since Γ is very sturdy, it is abstractly isomorphic to a product of finite simple groups, each of which is a simple group $\mathcal{D}(\mathcal{H}(\mathbb{F}_q))$ of Lie type with $l|q$. (See Lemma 4.1.3.) Any normal subgroup must just be a product of a subset of the factors, and so the quotient will be isomorphic to the product of the complementary factors, and hence not of l power order. This gives us the first bullet point in the definition of M -bigness.

Since Γ is very sturdy, it is assumed to act absolutely irreducibly, and the second bullet point in the definition of bigness follows.

In order to discuss the remaining points, we will let $m, \mathcal{G}_1, \dots, \mathcal{G}_m, k_1, \dots, k_m$ and r_1, \dots, r_m refer to the various objects of those names described in the definition of very sturdy, Definition 4.1.1. We can and will assume that these have been chosen with the additional useful properties as described in Remark 4.1.4. By definition, r_i is a faithful projective representation over k of $\pi_i(\mathcal{G}^{sc}(k_i))$; we will write V_i for this representation space¹, which must be absolutely irreducible since otherwise Γ would fail to act absolutely irreducibly, contradicting a hypothesis of sturdiness.

We now turn to proving the third bullet point in the definition of bigness. As explained in [Lar95a, 1.13], every k representation of $\mathcal{G}_i^{sc}(k_i)$ is a direct sum of irreducible representations over \bar{k} , and hence every self-extension of V_i is trivial. It follows that $H^1(\mathcal{G}_i^{sc}(k_i), \text{ad } V_i) = (0)$. From this and the fact that $\text{ad } V_i$ is semisimple (again from [Lar95a, 1.13]), so $\text{ad}^0 V_i$ is a direct summand, we see $H^1(\mathcal{G}_i^{sc}(k_i), \text{ad}^0 V_i) = (0)$. Then $H^1(\mathcal{G}_i^{sc}(k_i), \text{ad}^0 V_i \otimes_{k_i} k) = (0)$, and hence:

$$\begin{aligned}
& H^1\left(\prod_i \mathcal{G}_i^{sc}(k_i), \text{ad}^0 \bigotimes_i (V_i \otimes_{k_i} k)\right) \\
&= H^1\left(\prod_i \mathcal{G}_i^{sc}(k_i), \bigoplus_{S \subset \{1, \dots, m\}, S \neq \emptyset} \bigotimes_{i \in S} \text{ad}^0(V_i \otimes_{k_i} k)\right) \\
&= \bigoplus_{S \subset \{1, \dots, m\}, S \neq \emptyset} H^1\left(\prod_i \mathcal{G}_i^{sc}(k_i), \bigotimes_{i \in S} \text{ad}^0(V_i \otimes_{k_i} k)\right) \\
&= \bigoplus_{S \subset \{1, \dots, m\}, S \neq \emptyset} \left(\bigotimes_{i \in S} H^1(\mathcal{G}_i^{sc}(k_i), \text{ad}^0(V_i \otimes_{k_i} k)) \right) \otimes \left(\bigotimes_{i \notin S} \text{Hom}(\mathcal{G}_i^{sc}(k_i), k) \right) \\
&= \bigoplus_{S \subset \{1, \dots, m\}, S \neq \emptyset} (0) = (0)
\end{aligned}$$

But this tells us $H^1(\prod_i \mathcal{G}_i^{sc}(k_i), \text{ad } k^n) = (0)$, where $\prod_i \mathcal{G}_i^{sc}(k_i)$ acts on k^n via r .

Let K be the kernel of r . We have an exact sequence

$$1 \rightarrow K \rightarrow \prod_i \mathcal{G}_i^{sc}(k_i) \rightarrow r\left(\prod_i \mathcal{G}_i^{sc}(k_i)\right) \rightarrow 1$$

and hence an injection

$$H^1\left(r\left(\prod_i \mathcal{G}_i^{sc}(k_i)\right), (\text{ad}^0 k^n)^K\right) \hookrightarrow H^1\left(\prod_i \mathcal{G}_i^{sc}(k_i), \text{ad}^0 k^n\right)$$

and so since the group on the right vanishes, so does the group on the left. Since under r we have that K acts trivially on k^n , it acts trivially on $\text{ad}^0 k^n$. Thus $H^1(r(\prod_i \mathcal{G}_i^{sc}(k_i)), \text{ad}^0 k^n) = (0)$, and the third bullet point in the definition of

¹A convention which we will follow in the present proof is to use Roman letters like V for representation spaces of abstract finite groups, and cursive letters like \mathcal{V} for algebraic representations.

bigness holds for $r(\prod_i \mathcal{G}_i^{sc}(k_i))$. It follows that it holds for Γ , arguing as in [SW10, Prop 2.2].

All that remains is the fourth bullet point in the definition of bigness. Before we turn to this, we would first like to relate the r_i to algebraic representations. Fix some i , $1 \leq i \leq m$. We now apply Theorem 3.3.1 with:

- k there being our current k_i , and
- \mathcal{G} there being our current \mathcal{G}_i^{sc} ,

we see that $V_i \otimes \bar{k}$ (which is initially a representation of $\pi(\mathcal{G}_i(k_i))$, but can therefore be thought of as a representation of $\mathcal{G}_i(k_i)$ by composing with π) is of the form $\bigotimes_{j=0}^{[k:\mathbb{F}_l]-1} \mathcal{V}_{i,j}^{\text{Frob}^j}$, where each $\mathcal{V}_{i,j}$ is an irreducible algebraic representation of \mathcal{G}_i/k_i with highest weight λ satisfying $0 \leq \lambda(\mathbf{a}) \leq l-1$, and with k_i coefficients. By applying Corollary 3.3.3, we can bound the norm of $\mathcal{V}_{i,j}$, deducing that $\|\mathcal{V}_{i,j}\| < C_1(n)$, where $C_1(n)$ is the constant from Corollary 3.3.3.

Now, for each i , recall we have chosen (as per Remark 4.1.4) a Borel subgroup \mathcal{B}_i of \mathcal{G}_i defined over k_i , and a maximal torus \mathcal{T}_i for \mathcal{B}_i (automatically a maximal torus for \mathcal{G}_i). Let $\mathcal{V}_{i,j,0}$ be $\mathcal{V}_{i,j}^{\mathcal{U}_i}$, where \mathcal{U}_i is the unipotent radical of \mathcal{B}_i . Let $\lambda_{i,j} : \mathcal{T}_i \rightarrow \mathbb{G}_m$ give the action of \mathcal{T}_i on $\mathcal{V}_{i,j,0}$; as in the [SW10] (the second paragraph before the proof of Lemma 5.2 there), we see that $\lambda_{i,j}$ is a highest weight of $\mathcal{V}_{i,j}$ and occurs as a weight with multiplicity 1.

Let $e_{i,j}$ be a vector in $\mathcal{V}_{i,j}$, then let $e_i = \bigotimes_j e_{i,j}$, a vector in $\bigotimes_j \mathcal{V}_{i,j}$. Since V_i corresponds to $\bigotimes_j \mathcal{V}_{i,j}^{\text{Frob}^j}$ under the correspondence of Theorem 3.3.1, and in particular they have the same underlying vector space over k , we can think of this e_i also as a vector in V_i . Finally let $e = \bigotimes_i e_i$, a vector in a representation space for r , V say. We see, by Lemma 4.3.5 that we can find an element $g \in \prod_i \mathcal{G}_i(k_i)$ such that e is an eigenvector of g whose corresponding eigenvalue α is a simple root of the characteristic polynomial of $g|V$, and which indeed has the property that any other root β of this polynomial has $\alpha^M \neq \beta^M$.

On the other hand, by [SW10, Lemma 5.2], and the arguments immediately before we see that every nonzero irreducible submodule of $\text{ad}(\mathcal{V}_{i,j})$ has non-zero projection onto $\text{ad}(\mathcal{V}_{i,j,0})$; whence every nonzero irreducible submodule of $\text{ad}(\bigotimes_j \mathcal{V}_{i,j}^{\text{Frob}^j})$ has nonzero projection onto $\text{ad}(\bigotimes_j \mathcal{V}_{i,j,0}^{\text{Frob}^j})$; and thus (using Theorem 3.3.1, Theorem 3.3.2, and) we see that every nonzero irreducible submodule of $V_i \otimes \bar{k}$ has nonzero projection onto $\langle e_i \rangle \otimes \bar{k}$. It follows every nonzero irreducible submodule of V_i has nonzero projection onto $\langle e_i \rangle \otimes \bar{k}$. Thus every nonzero irreducible submodule of V has nonzero projection onto $\langle e \rangle$. This is as required. \square

REF

Corollary 4.4.2. *Let M and n be positive integers. There is a constant $C_3(M, n)$ depending only on M and n with the following property: if l is a prime number which is larger than $C_3(M, n)$, k/\mathbb{F}_l a finite extension, and $\Gamma < \text{GL}_n(k)$ a sturdy subgroup, then Γ is M -big.*

Proof. We take the constant $C_3(M, n)$ to be as in the Proposition. Then, given a sturdy Γ , we look at the chain of subgroups $\Gamma_1 \triangleleft \Gamma_2 \triangleleft \cdots \triangleleft \Gamma_r = \Gamma$. We apply the Proposition to see that Γ_1 is M -big; we then apply Proposition 2.1.2 to see inductively that each Γ_i , $i > 1$ is M -big. In particular, $\Gamma = \Gamma_r$ is M -big. \square

5. THE MAIN RESULT

5.1. The heart of what we will prove is the following rather technical proposition.

Proposition 5.1.1. *For each positive integer n , there is an integer A_n with the following property. Let $l > A_n$ be a prime, k/\mathbb{F}_l a finite extension with $l \nmid [k : \mathbb{F}_l]$, and $\Gamma < \mathrm{GL}_n(k)$ a subgroup containing k^\times . For convenience of notation, let us also choose a number field L and prime λ of L such that $\mathcal{O}_L/\lambda\mathcal{O}_L = k$. Then one of the following must occur:*

- (1) Γ does not act absolutely irreducibly on k^n .
- (2) Γ lies inside some imprimitive subgroup $G_{V_1 \oplus \dots \oplus V_m}$ (see Definition 3.1.1); that is, Γ preserves a direct sum decomposition, $k^n = V_1 \oplus \dots \oplus V_m$ (where $\dim V_i = n/m$ for all i), though it need not preserve the individual terms in the direct sum.
- (3) We can find a k vector space V_k , an L_λ vector space V_L , an \mathcal{O}_{M_λ} lattice $\Lambda \subset V_L$, a finite subgroup $G < \mathrm{GL}(\Lambda)$, and an isomorphism $k^n \cong V_k \otimes \overline{V_L}$ (where $\overline{V_L}$ is the k -vector space $\Lambda \otimes_{\mathcal{O}_{L_\lambda}} k$), such that we can factor the map $\Gamma \hookrightarrow \mathrm{GL}(k^n) \rightarrow \mathrm{PGL}(k^n)$ through the map

$$\mathrm{PGL}(V_k) \times G \hookrightarrow \mathrm{PGL}(V_k) \times \mathrm{GL}(\Lambda) \twoheadrightarrow \mathrm{PGL}(V_k) \times \mathrm{PGL}(\Lambda) \twoheadrightarrow \mathrm{PGL}(V_k) \times \mathrm{PGL}(\overline{V_L}) \hookrightarrow \mathrm{PGL}(k^n)$$

- (4) Γ is a sturdy subgroup, in the sense of Definition 4.1.2.

Furthermore, in case (3) the order of the group G can be bounded in terms of n .

Proof. We prove the claim by induction on n . We may therefore inductively suppose the existence of A_i for all $i < n$.

By applying Proposition 3.2.1 with \mathcal{G} being the algebraic group \mathcal{PGL} , we see that there exists a constant X and a finite set S of almost simple groups, (a group H is *almost simple* if $G < H < \mathrm{Aut} G$ for some simple group G) such that for all $l > X$ and finite extensions k/\mathbb{F}_l we have that

- Every subgroup of $\mathrm{PGL}(\mathbb{F}_l)$ which is a finite almost simple group is either isomorphic to a member of S or to an almost simple group whose corresponding simple group is a derived group of an adjoint group of Lie type, $\mathcal{D}(\mathcal{H}(\mathbb{F}_q))$, where $l|q$ and $\mathcal{D}(\mathcal{H}(\mathbb{F}_q)) = \mathrm{Im}(\mathcal{H}^{sc}(\mathbb{F}_q) \rightarrow \mathcal{H}(\mathbb{F}_q))$. (5.1.1)

We can and do choose A_n to be large enough that, if l is a prime larger than A_n :

- $l > X$. (5.1.2)
- $l > A_i$, $i = 1, \dots, n-1$. (5.1.3)
- If $n = p^m$ is an odd prime power, then l is coprime to $\#p_+^{1+2m}$ and to $\#\mathrm{Sp}_{2m}(\mathbb{F}_p)$. (5.1.4)
- If $n = 2^m$ is a power of 2, then $l > 2$ (and hence is coprime $\#2_+^{1+2m}$ and $\#2_-^{1+2m}$), and furthermore l is coprime to both $\#\mathrm{GO}_{2m}^+(\mathbb{F}_2)$ and $\#\mathrm{GO}_{2m}^-(\mathbb{F}_2)$. (5.1.5)
- l is coprime to the orders of all the groups in the set S . (5.1.6)
- We have $l \nmid m$ for all $m \leq n$. (5.1.7)

We must now show that this A_n will have the property described in the proposition.

We consider first the special case where Γ is all of $\mathrm{GL}_n(k)$. In this case Γ contains $\mathrm{SL}_n(k)k^\times$ as a normal subgroup with prime-to- l index, while $\mathrm{SL}_n(k)k^\times$ is clearly very sturdy, taking \mathcal{G} as the restriction of scalars of \mathcal{SL}_n from k to \mathbb{F}_l (this is manifestly simply connected and semisimple), and the obvious map $\mathcal{G} \rightarrow \mathcal{SL}_{n[k:\mathbb{F}_l]}$ (which is obviously algebraic). Thus Γ is in this case sturdy.

Otherwise, we apply the Aschbacher-Dynkin Theorem (Theorem 3.1.2), to Γ . We see that Γ is contained in some subgroup of $\mathrm{GL}_n(k)$ of one of the kinds described in

Definition 3.1.1, (1)–(6). The rest of our proof will be broken into cases, depending on in which kind of subgroup Γ is contained.

- *Case 1: A reducible subgroup.* In this case, we see that $\pi^{-1}(\Gamma)$ acts reducibly, and we are done (alternative 1 of the theorem to be proved).
- *Case 2: An imprimitive subgroup, $G_{V_1 \oplus \dots \oplus V_m}$.* In this case, we are also done (alternative 2 of the theorem to be proved).
- *Case 3: The stabilizer of a binary tensor product decomposition, factor by factor.* Suppose Γ stabilizes a tensor product decomposition $V_1 \otimes V_2$.

In this case, the map $\Gamma \hookrightarrow \mathrm{GL}_n(k) \rightarrow \mathrm{PGL}_n(k)$ will factor through the obvious map $\mathrm{PGL}(V_1) \times \mathrm{PGL}(V_2) \hookrightarrow \mathrm{PGL}(k^n)$, giving a map $\phi : \Gamma \rightarrow \mathrm{PGL}(V_1) \times \mathrm{PGL}(V_2)$. Let $\pi_i : \mathrm{PGL}(V_1) \times \mathrm{PGL}(V_2) \rightarrow \mathrm{PGL}(V_i)$ ($i=1,2$) denote the projection, and let $q_i : \mathrm{GL}(V_i) \rightarrow \mathrm{PGL}(V_i)$ ($i=1,2$) denote the natural quotient map. Finally, let $\Gamma_i := q_i^{-1}(\pi_i^{-1}(\phi_i(\Gamma)))$.

By point 5.1.3, $l > A_{\dim V_1}$, and we may apply our inductive hypothesis to yield that Γ_1 satisfies one of the alternatives (1–4) in the statement of the proposition. We first show that we are done if any of the first three alternatives hold.

- *Case 3a: Γ_1 does not act absolutely irreducibly.* In this case, after an extension of fields k'/k , $V_1 \otimes_k k'$ has a nontrivial proper subspace W which is Γ_1 stable. Then $(k')^n$ has a nontrivial proper subspace $W \otimes_{k'} (V_2 \otimes_k k')$ which is Γ stable, and we have alternative 1 in the statement of the present proposition.
- *Case 3b: Γ_1 acts imprimitively.* In this case, Γ_1 will preserve a direct sum decomposition $V_1 = W_1 \oplus \dots \oplus W_m$ for some m , where $\dim W_i = \dim V_1/m$ for all i . Then Γ will preserve the direct sum decomposition $V_1 \otimes V_2 \oplus \dots \oplus W_m \otimes V_2$, and we have alternative 2 in the statement of the present proposition.
- *Case 3c: The map $\Gamma_1 \hookrightarrow \mathrm{GL}(V_1) \rightarrow \mathrm{PGL}(V_1)$ factors through*

$$\begin{aligned} \mathrm{PGL}(V_{1,k}) \times G &\hookrightarrow \mathrm{PGL}(V_{1,k}) \times \mathrm{GL}(\Lambda) \rightarrow \mathrm{PGL}(V_{1,k}) \times \mathrm{PGL}(\Lambda) \\ &\twoheadrightarrow \mathrm{PGL}(V_{1,k}) \times \mathrm{PGL}(\overline{V_{1,L}}) \hookrightarrow \mathrm{PGL}(V_1). \end{aligned}$$

(Here we have a k vector space V_k , an L_λ vector space V_L , an \mathcal{O}_{L_λ} lattice $\Lambda \subset V_L$, a finite subgroup $G < \mathrm{GL}(\Lambda)$, and an isomorphism $k^n \cong V_k \otimes \overline{V_L}$.) In this case, we have $V \cong V_1 \otimes V_2 \cong V_k \otimes \overline{V_{1,L}}$ where $V_k := V_2 \otimes V_{1,k}$, and a commutative diagram as in Figure 1.

Thus we have alternative 3 in the statement of the present proposition.

Thus we have reduced to the case where alternative (4) in the statement of the Proposition holds for Γ_1 . We may similarly reduce to the case where (4) holds for Γ_2 . That is, we see that Γ_1 and Γ_2 are both sturdy. We may also assume that Γ acts absolutely irreducibly (else we are done, via the first alternative). By applying Lemma 4.2.1, we see that Γ is sturdy, and we have the fourth alternative in the proposition to be proved.

- *Case 4: The stabilizer of a tensor product decomposition, $k^n = V_1 \otimes \dots \otimes V_m$ not necessarily factor by factor.* In this case, it is easy to see that for each $\gamma \in \Gamma$, γ determines a permutation of the V_i , and we have a homomorphism $\phi : \Gamma \rightarrow S_X$, where $X = \{V_1, \dots, V_m\}$; that is, Γ acts on X . If this action is not transitive, then we may write V as a tensor product with Γ preserving

think of $\mathbb{P}V_{1,k}$ as a projective representation of Γ_1 ; and that moreover, if we think of V_1 as a representation of Γ_1 in the obvious way, then the associated projective representation satisfies $\mathbb{P}V_1 \cong \mathbb{P}\overline{V_{1,L}} \otimes \mathbb{P}V_{1,k}$. Then we see

$$\begin{aligned} \mathbb{P}V &\cong \mathbb{P}(\otimes\text{-Ind}_{\Gamma}^{\Gamma_1} V_1) \cong \otimes\text{-Ind}_{\Gamma}^{\Gamma_1} \mathbb{P}V_1 \\ &\cong \otimes\text{-Ind}_{\Gamma}^{\Gamma_1} \mathbb{P}\overline{V_{1,L}} \otimes \mathbb{P}V_{1,k} \\ &\cong (\otimes\text{-Ind}_{\Gamma}^{\Gamma_1} \mathbb{P}\overline{V_{1,L}}) \otimes (\otimes\text{-Ind}_{\Gamma}^{\Gamma_1} \mathbb{P}V_{1,k}) \\ &\cong \overline{\mathbb{P}\otimes\text{-Ind}_{\Gamma}^{\Gamma_1} V_{1,L}} \otimes (\otimes\text{-Ind}_{\Gamma}^{\Gamma_1} \mathbb{P}V_{1,k}) \\ &\cong \mathbb{P}\overline{V_L} \otimes \mathbb{P}V_k \end{aligned}$$

Here $V_L := \otimes\text{-Ind}_{\Gamma}^{\Gamma_1} V_{1,L}$ is a characteristic 0 representation of Γ , with an invariant lattice $\Lambda^{\otimes m}$ and $\mathbb{P}V_k := \otimes\text{-Ind}_{\Gamma}^{\Gamma_1} \mathbb{P}V_{1,k}$ is a characteristic l projective representation. These furnish us with a map $\Gamma \rightarrow \text{GL}(\Lambda^{\otimes m})$ (with finite image, G say) and a map $\Gamma \rightarrow \text{PGL}(V_{1,k})$. The isomorphisms above then tell us that we can factor $\Gamma \hookrightarrow \text{GL}(V) \twoheadrightarrow \text{PGL}(V)$ through

$$\text{PGL}(V_k) \times G \hookrightarrow \text{PGL}(V_k) \times \text{GL}(\Lambda^{\otimes m}) \twoheadrightarrow \text{PGL}(V_k) \times \text{PGL}(\overline{V_L}) \hookrightarrow \text{PGL}(V).$$

Thus we have alternative 3 in the statement of the present proposition.

- *Case 4d: Γ_1 is sturdy.* In this case, we introduce a further subgroup $N := \ker \phi$ of Γ , the subgroup acting trivially on X . N is normal in Γ , with index dividing $m!$, and so if we put $N_1 = q^{-1}(\pi(N \cap \text{Stab}_{\Gamma} V_1)) = q^{-1}(\pi(N))$, then N_1 is normal in Γ_1 with index dividing $m!$. In particular, by point 5.1.7, and the obvious fact that $m < n$, we see $(l, m) = 1$, so by Lemma 4.2.2, we see that N_1 is sturdy.

We can construct, by analogy with Γ_1 , further subgroups $\Gamma_2, \dots, \Gamma_m$. Since, given any element γ_1 of Γ_1 , we can find an element of Γ_2 which acts on V_2 in the same way as γ_1 acted on V_1 (by conjugating by an element of Γ which acts to move V_1 to V_2 , possible since the action of Γ on X is transitive), we see $\Gamma_2, \dots, \Gamma_m$ are also big. We construct N_2, \dots, N_m by analogy with N_1 above, and we see that each is sturdy, by the same argument. It follows from Lemma 4.2.1, applied repeatedly, that N is sturdy. Then since $N \triangleleft \Gamma$ with index prime to l (since dividing $m!$ —see point 5.1.7), we conclude Γ is sturdy.

Thus we are done, having the fourth alternative in the proposition to be proved.

- *Case 5: A subgroup of extraspecial type, $G_{p^{1+2m}}$.* Let us first suppose $p \neq 2$. Let G be the subgroup of $\text{GL}_n(\overline{\mathbb{Q}})$ isomorphic to p_+^{1+2m} as constructed in §3.10.2 of [Wil09], N its normalizer in $\text{GL}_n(\overline{\mathbb{Q}})$ (so $N \cong p_+^{1+2m} \rtimes \text{Sp}_{2m}(\mathbb{F}_p)$.) Let \bar{G} and \bar{N} be their reductions mod l , subgroups of $\text{GL}_n(k)$. We have that $\#\bar{G} \mid \#G_{p^{1+2m}}$ and $\#G_{p^{1+2m}} \mid \#\bar{N} \#k^{\times}$, while $\#\bar{N} \mid \#N = \#p_+^{1+2m} \rtimes \text{Sp}_{2m}(\mathbb{F}_p)$. Thus

$$\#\bar{G} \mid (\#k^{\times}) (\#p_+^{1+2m}) (\#\text{Sp}_{2m}(\mathbb{F}_p))$$

and since on the other hand $1 = (\#k^{\times}, l) = (p, l) = (\#\text{Sp}_{2m}(\mathbb{F}_p), l)$ (by point 5.1.4), we see that $(\#\bar{G}, l) = 1$. We may think of $\Gamma \hookrightarrow \text{GL}_n(k)$ as

being a characteristic l representation of Γ ; since $(\#\Gamma, l) = 1$, this lifts to a characteristic zero representation $r : \Gamma \rightarrow \mathrm{GL}(\Lambda)$ where $\Lambda \subset L_\lambda^n$ is a lattice.

We then have alternative 3 of the present proposition, taking $V_k = k$, $G = r(\Gamma)$, and mapping $\Gamma \rightarrow \mathrm{PGL}(V_k) \times G$ via $(1, r)$, where 1 is the constant function taking value the identity.

The case that $p = 2$ is completely analogous, using 5.1.5.

- *Case 6: Γ/k^\times is contained in the image of an injective homomorphism $\phi : H \hookrightarrow \mathrm{PGL}_n(k)$, where H is an almost simple group.*

We first consider the case where H is isomorphic to one of the groups in the set S constructed at the beginning of the proof. In such a case, say $\Gamma/k^\times \cong G_0$, $G_0 \in S$, $\#\Gamma = \#(\Gamma/k^\times) \#k^\times = \#G_0 \#k^\times$, so since $(\#k^\times, l) = (\#G_0, l) = 1$ (using point 5.1.6 above), we have $(\#\Gamma, l) = 1$. Thus, thinking of the inclusion $\Gamma \hookrightarrow \mathrm{GL}_n(k)$ as a representation of Γ in characteristic l , we can lift the representation to characteristic zero, and hence deduce that alternative 3 of the Proposition holds, just as in case 5 analyzed above.

Thus we may assume on the one hand that Γ/k^\times is isomorphic to an almost simple group H ; and on the other hand, this almost simple group cannot be isomorphic to any group in the set S . But then, given point 5.1.1 above, we see that Γ/k^\times is isomorphic to an almost simple group whose corresponding simple group is a derived group of an adjoint group of Lie type, $\mathcal{D}(\mathcal{H}(\mathbb{F}_q))$, where $l|q$ and $\mathcal{D}(\mathcal{H}(\mathbb{F}_q)) = \mathrm{Im}(\mathcal{H}^{sc}(\mathbb{F}_q) \rightarrow \mathcal{H}(\mathbb{F}_q))$.

This tells us that H has a normal subgroup, N say, where $N \cong \mathcal{D}(\mathcal{H}(\mathbb{F}_q))$.

Claim. *The degree $[\mathbb{F}_q : \mathbb{F}_l]$ divides $[k : \mathbb{F}_l]$.*

Proof. We can consider $r : \mathcal{H}^{sc}(\mathbb{F}_q) \rightarrow \mathcal{H}(\mathbb{F}_q) \xrightarrow{\sim} N \hookrightarrow H \hookrightarrow \mathrm{PGL}_n(k)$ as a projective representation of the abstract group $\mathcal{H}^{sc}(\mathbb{F}_q)$. We can then extend the coefficients of this representation to $\bar{\mathbb{F}}_l$. Applying Theorem 3.3.1, we see that, after extending coefficients in this way, it can be constructed as the restriction to \mathbb{F}_q points of a product of Frobenius twists of algebraic representations, as described in the statement of that theorem. But considering representations of this form, we immediately see that if $[\mathbb{F}_q : \mathbb{F}_l] \nmid [k : \mathbb{F}_l]$, we will not be able to conjugate our $\bar{\mathbb{F}}_l$ representation to a representation defined over k , which is a contradiction, because our $\bar{\mathbb{F}}_l$ representation came from one defined over k . \square

Since, by assumption, $l \nmid [k : \mathbb{F}_l]$ we see that $l \nmid [\mathbb{F}_q : \mathbb{F}_l]$, and hence (using the complete description of the outer automorphism groups of groups of Lie type given on page xv of [CCN⁺]) that $\mathrm{Out} N$ has order prime to N . Thus H/N has order prime to l . Let $q : \mathrm{GL}_n(k) \twoheadrightarrow \mathrm{PGL}_n(k)$ denote the quotient map, and let $\Gamma_2 = \Gamma = q^{-1}(\phi(H))$ and $\Gamma_1 = q^{-1}(\phi(N))$. Then we see that $\Gamma_1 \triangleleft \Gamma_2$ with $\#(\Gamma_2/\Gamma_1) = \#(H/N)$ prime to l . So it suffices to prove that Γ_1 is sturdy (since then Γ is sturdy, so we are done by the fourth alternative in the statement of the proposition).

But $\mathrm{Im}(\mathcal{H}^{sc}(\mathbb{F}_q) \rightarrow \mathcal{H}(\mathbb{F}_q))$ is the simple group $\mathcal{D}(\mathcal{H}(\mathbb{F}_q))$, so Γ_1 is immediately seen to be very sturdy, taking in the definition $m = 1$, $k_1 = \mathbb{F}_q$, $\mathcal{G} = \mathcal{H}^{sc}$, and $r_1 : \mathcal{D}(\mathcal{H}(\mathbb{F}_q)) \xrightarrow{\sim} N \hookrightarrow \mathrm{PGL}_n(k)$.

This completes the proof of the proposition. (The last sentence of the proposition may be straightforwardly verified inductively.) \square

We can now reformulate our proposition in the language of representations.

Proposition 5.1.2. *For each positive integer n , there is an integer A_n with the following property. Let $l > A_n$ be a prime, k/\mathbb{F}_l a finite extension with $l \nmid [k : \mathbb{F}_l]$, Γ_0 a group, and $r : \Gamma_0 \rightarrow \mathrm{GL}_n(k)$ a representation. For convenience of notation, let us choose a number field L and prime λ of L such that $\mathcal{O}_L/\lambda\mathcal{O}_L = k$. Then one of the following must occur:*

- (1) r does not act absolutely irreducibly on k^n ,
- (2) there is a proper subgroup $\Gamma'_0 < \Gamma_0$ and representation $r' : \Gamma'_0 \rightarrow \mathrm{GL}_m(k)$ such that $r = \mathrm{Ind}_{\Gamma'_0}^{\Gamma_0} r'$,
- (3) there are representations $r_1 : \Gamma_0 \rightarrow \mathrm{GL}_m(\mathcal{O}_L)$ and $r_2 : \Gamma_0 \rightarrow \mathrm{GL}_{m'}(k)$ with open kernels and with $m > 1$ such that $r = \bar{r}_1 \otimes r_2$, or
- (4) $k^\times r(\Gamma_0)$ is a sturdy subgroup.

Furthermore, in case (3) the order of the image of r_1 can be bounded in terms of n .

Proof. We will take the constants A_n to be the same constants as in the previous proposition, and will show that the property we require holds with this choice of the A_n . To see this, we apply the previous proposition to $r(\Gamma_0)k^\times < \mathrm{GL}_n(k)$. We deduce that one of the four possibilities (1)–(4) in the conclusion of that proposition must hold. We split the remainder of our proof into cases, according to which of the alternatives hold

- *Case 1:* $k^\times r(\Gamma_0)$ does not act absolutely irreducibly. In this case we immediately see that the first alternative of the present Proposition holds.
- *Case 2:* $k^\times r(\Gamma_0) \subset G_{V_1 \oplus \dots \oplus V_m}$, for some direct sum decomposition $V \cong V_1 \oplus \dots \oplus V_m$. In this case, for each $\gamma \in \Gamma_0$, $r(\gamma)$ must send each V_i into some other V_i , and indeed will determine in this way a permutation of the V_i . Thus Γ_0 acts on the set $\{V_i | 1 \leq i \leq m\}$. If this permutation action is not transitive, then $\bigoplus_{V_i \in \mathcal{O}_{\Gamma_0} V_1} V_i$ is a nontrivial Γ_0 submodule of V , so Γ_0 acts irreducibly, and see that the first alternative of the present proposition holds. So we assume the action is transitive. Then let Γ'_0 be the stabilizer of V_1 . $r|_{\Gamma'_0}$ sends V_1 to itself, and we then get a representation $r' : \Gamma'_0 \rightarrow \mathrm{GL}(V_1)$. We then see that $r = \mathrm{Ind}_{\Gamma'_0}^{\Gamma_0} r'$.
- *Case 3:* the projective representation factors through a tensor product, and the image on one tensor factor lifts to a characteristic zero representation. Then see that the other tensor factor in fact lifts from being a projective representation to an ordinary representation. This gives us what we need. (The bound on the size of G gives us the bound on the size of the image of r_1 .)
- *Case 4:* $k^\times r(\Gamma)$ is sturdy. In this case we immediately have the fourth alternative of the present Proposition.

This completes the proof. \square

Finally, we can deduce the main theorem (Theorem 1.1.1 in the introduction).

Theorem 5.1.3. *For each pair of positive integers n and M , there is an integer $C(M, n)$ with the following property. Let $l > C(M, n)$ be a prime, k/\mathbb{F}_l a finite extension with $l \nmid [k : \mathbb{F}_l]$, Γ_0 a group, and $r : \Gamma_0 \rightarrow \mathrm{GL}_n(k)$ a representation. For convenience of notation, let us choose a number field L and prime λ of L such that*

$\mathcal{O}_L/\lambda\mathcal{O}_L = k$. Suppose that the image of r is not M -big. Then one of the following must hold:

- (1) r does not act absolutely irreducibly on k^n ,
- (2) there is a proper subgroup $\Gamma'_0 < \Gamma_0$ and representation $r' : \Gamma'_0 \rightarrow \mathrm{GL}_m(k)$ such that $r = \mathrm{Ind}_{\Gamma'_0}^{\Gamma_0} r'$, or
- (3) there are representations $r_1 : \Gamma_0 \rightarrow \mathrm{GL}_m(\mathcal{O}_L)$ and $r_2 : \Gamma_0 \rightarrow \mathrm{GL}_{m'}(k)$ with open kernels and with $m > 1$ such that $r = \bar{r}_1 \otimes r_2$.

Furthermore, in case (3) the order of the image of r_1 can be bounded in terms of n .

Proof. This follows immediately from Proposition 5.1.2, Proposition 2.1.3, and Corollary 4.4.2. \square

We also now prove the Lemma from the introduction asserting that the third option above never occurs for the residual representations of regular crystalline Galois representations where l is large compared to the weight.

Proof of Lemma 1.1.4. Suppose that n, N, F and a are given as in the statement of the lemma. We choose $D(n, N, a) = (3(\max_{\tau,i} a_{\tau,i} - \min_{\tau,i} a_{\tau,i}) + 2)N!$.

Now suppose that l, L, ρ , and ι are as in the lemma. Remember that l has been chosen not to ramify in F , so that we may apply Fontaine-Laffaille theory. Choose v to be some place above l in F . There is some unramified local extension K/F_v such that the semisimplification of $\rho|_{I_K}$ breaks up as a direct sum of characters, say χ_1, \dots, χ_n . Let us choose an embedding $\sigma : K \hookrightarrow \bar{\mathbb{Q}}_l$. We may then write

$$\chi_i = \omega_{[L:\mathbb{Q}_l]}^{c_{i,0}+c_{i,1}l+c_{i,2}l^2+\dots+c_{i,[L:\mathbb{Q}_l]-1}l^{[L:\mathbb{Q}_l]-1}}$$

where the $c_{i,j}$ are integers and $\omega_{[L:\mathbb{Q}_l]}$ is Serre's fundamental character of niveau $[L:\mathbb{Q}_l]$, and moreover

$$\{c_{1,t}, c_{2,t}, \dots, c_{n,t}\} = a_{\iota \circ \mathrm{Frob}^t \circ \sigma|_L}$$

Write k' for the residue field of L . We may choose an element $\gamma \in I_K \subset G_F$ which maps to a generator $g \in (k')^\times$ under $\omega_{[L:\mathbb{Q}_l]}$.

Now, suppose for contradiction that $\bar{\rho}$ did break up as a tensor product $\bar{\rho}' \otimes r''$. Choose distinct \bar{k} eigenvectors of $\bar{\rho}'(\gamma)$, say $e_{\bar{\rho}',1}$ and $e_{\bar{\rho}',2}$. The ratio of their eigenvalues must be α , an r th root of unity for some $r \leq N!$. (This is since all eigenvalues of $\rho(\gamma)$ are roots of unity of order dividing the order of the image of ρ .) Choose also a \bar{k} eigenvector of $r''(\gamma)$, say $e_{r''}$. Then $e_{\bar{\rho}',1} \otimes e_{r''}$ and $e_{\bar{\rho}',2} \otimes e_{r''}$ are eigenvectors of $\bar{\rho}(\gamma) = (\bar{\rho}' \otimes r'')(\gamma)$. The ratio of their eigenvalues is again α ; but on the other hand we see that the ratio of their eigenvalues is $\chi_i(\gamma)/\chi_j(\gamma)$ for some i, j , $1 \leq i, j \leq n$, with i, j distinct.

That is

$$\begin{aligned} \alpha &= \chi_i(\gamma)/\chi_j(\gamma) = g^{c_{i,0}+c_{i,1}l+\dots+c_{i,[L:\mathbb{Q}_l]-1}l^{[L:\mathbb{Q}_l]-1}}/g^{c_{j,0}+c_{j,1}l+\dots+c_{j,[L:\mathbb{Q}_l]-1}l^{[L:\mathbb{Q}_l]-1}} \\ &= g^{(c_{i,0}-c_{j,0})+l(c_{i,1}-c_{j,1})+l^2(c_{i,2}-c_{j,2})+\dots+l^{[L:\mathbb{Q}_l]-1}(c_{i,[L:\mathbb{Q}_l]-1}-c_{j,[L:\mathbb{Q}_l]-1})} \end{aligned}$$

So, since α is an r th root of unity and g is a generator,

(5.1.8)

$$r \left[(c_{i,0} - c_{j,0}) + l(c_{i,1} - c_{j,1}) + \dots + l^{[L:\mathbb{Q}_l]-1}(c_{i,[L:\mathbb{Q}_l]-1} - c_{j,[L:\mathbb{Q}_l]-1}) \right] \equiv 0 \pmod{l^{[L:\mathbb{Q}_l]} - 1}$$

On the other hand, we have the following elementary fact, whose proof is an exercise:

Claim. *Since $r < N!$, and $l > (3(\max_{\tau,i} d_{\tau,i} - \min_{\tau,i} d_{\tau,i}) + 2)N!$, if we write S for the set $\mathbb{Z} \cap [\min_{\tau,i} d_{\tau,i} - \max_{\tau,i} d_{\tau,i}, \max_{\tau,i} d_{\tau,i} - \min_{\tau,i} d_{\tau,i}]$, then the map*

$$S \times \cdots \times S \rightarrow \mathbb{Z}/(l^{[L:\mathbb{Q}_l]} - 1)\mathbb{Z}$$

$$(b_0, \dots, b_{[L:\mathbb{Q}_l]-1}) \mapsto (b_0 + lb_1 + \cdots + l^{[L:\mathbb{Q}_l]-1}b_{[L:\mathbb{Q}_l]-1})r$$

is injective.

Thus, since each $c_{i,t}$ is an $a_{\tau,i}$ for some τ and i , we see that $c_{i,t} - c_{j,t} \in S$ for each i, j, t , so the claim and eq 5.1.8 tell us:

$$c_{i,0} - c_{j,0} = c_{i,1} - c_{j,1} = \cdots = c_{i,[L:\mathbb{F}_l]-1} - c_{j,[L:\mathbb{F}_l]-1} = 0$$

But the fact that $c_{i,0} = c_{j,0}$ tells us some two members of $a_{i \circ \sigma|_L}$ are equal, which is contrary to hypothesis. The lemma is therefore proved. \square

5.2. Given Theorem 5.1.3, it is perhaps natural to ask how often we can expect representations of the forms (1)–(3) to have big image. (That is, how often representations the theorem does not guarantee to have big image have big image nonetheless.) We will make only some very superficial remarks. First, all representations of type (1) will fail to be big, since acting absolutely irreducibly is part of the definition of bigness.

Second, it seems that representations of the form (3) will ‘for the most part’ fail to have big image. For instance, an inspection of $[\text{CCN}^+]$ shows that the vast majority of characteristic 0 representations of finite simple nonabelian groups G seem to have dimension n much larger than the maximal order m of an element in G , and in such a case (since the n roots of characteristic polynomials of the elements of G acting via the representation must then be k th roots of unity where $k \leq m$) it seems perhaps a little unlikely that one would be able to find an element in G whose characteristic polynomial has a simple root. Some rudimentary computer calculations seem to back up this intuition.

On the other hand, for very low dimensional representations, it seems that such reductions of Artin representations do ‘generally’ have big image. For instance, in §4 of [BLGG10] it is shown for $l > 5$ that any two dimensional representation which acts absolutely irreducibly and is not induced is in fact 2-big, and the author believes that any three dimensional representation which is not induced and acts absolutely irreducibly is both 2-big and 3-big. (For four dimensional representations, things rapidly become less nice: if we let $\rho : (2.A_4) \times (2.A_4) \rightarrow \text{GL}_4(\overline{\mathbb{Q}})$ by tensoring together two copies of the standard two-dimensional representation of $2.A_4$, it is not hard to see that the image of $\bar{\rho}$ fails to be big. Here $2.A_4$ is the binary tetrahedral group, the unique double cover of A_4 .)

This leaves representations of the form (2), about which we have less to say. The question of whether an induced representation has big image seems to be rather delicate, and depends on precise details of the induced representation. It is certainly possible for such a representation to fail to be big (for instance, let K_1/\mathbb{Q} and K_2/\mathbb{Q} be two quadratic extensions, k/\mathbb{F}_l a finite extension, $\theta_1 : G_{K_1} \rightarrow k^\times$ and $\theta_2 : G_{K_2} \rightarrow k^\times$ two characters, and $r = (\text{Ind}_{G_{K_1}}^{G_{\mathbb{Q}}} \theta_1) \otimes (\text{Ind}_{G_{K_1}}^{G_{\mathbb{Q}}} \theta_2)$; then it is relatively straightforward to show that r will fail to have big image). On the other hand, it seems that many such induced representations will in fact have big image.

REFERENCES

- [Asc84] M. Aschbacher, *On the maximal subgroups of the finite classical groups*, *Inventiones mathematicae* **76** (1984), no. 3, 469–514.
- [BLGG09] Tom Barnet-Lamb, Toby Gee, and David Geraghty, *The Sato-Tate Conjecture for Hilbert Modular Forms*, Preprint, 2009.
- [BLGG10] ———, *Congruences between Hilbert modular forms: constructing ordinary lifts in parallel weight two*, Preprint, 2010.
- [BLGGT10] Tom Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor, *Potential automorphy and change of weight*, Preprint, 2010.
- [BLGHT09] Tom Barnet-Lamb, David Geraghty, Michael Harris, and Richard Taylor, *A family of Calabi-Yau varieties and potential automorphy II*, Preprint, 2009.
- [CCN⁺] JH Conway, RT Curtis, SP Norton, RA Parker, and RA Wilson, *Atlas of finite groups*, Bulletin of the London Mathematical Society.
- [CHT08] Laurent Clozel, Michael Harris, and Richard Taylor, *Automorphy for some l -adic lifts of automorphic mod l Galois representations*, *Pub. Math. IHES* **108** (2008), 1–181.
- [Dyn52] E.B. Dynkin, *Maximal subgroups of the classical groups (Russian)*, *Trudy Moskov. Mat. Obšč* **1** (1952), 39–166.
- [Dyn00] EB Dynkin, *Maximal subgroups of classical groups*, *Selected papers of EB Dynkin with commentary* **4** (2000), 37.
- [Ger09] David Geraghty, *Modularity lifting theorems for ordinary Galois representations*, preprint, 2009.
- [HSBT06] Michael Harris, Nick Shepherd-Barron, and Richard Taylor, *A family of Calabi-Yau varieties and potential automorphy*, Preprint, 2006.
- [HW54] G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers*, New York: Oxford (1954).
- [Jan97] J.C. Jantzen, *Low dimensional representations of reductive groups are semisimple*, *Algebraic groups and Lie groups: a volume of papers in honour of the late RW Richardson* (1997), 255.
- [Lar95a] M. Larsen, *Maximality of Galois actions for compatible systems*, *Duke Mathematical Journal* **80** (1995), no. 3, 601–630.
- [Lar95b] ———, *On the semisimplicity of low-dimensional representations of semisimple groups in characteristic p* , *Journal of Algebra* **173** (1995), no. 2, 219–236.
- [Ste63] R. Steinberg, *Representations of algebraic groups*, *Nagoya Math. J* **22** (1963), 33–56.
- [Ste68] ———, *Endomorphisms of linear algebraic groups*, *Amer Mathematical Society*, 1968.
- [SW10] A. Snowden and A. Wiles, *Bigness in compatible systems*, *Arxiv preprint arXiv:0908.1991* (2010).
- [Whi10] Paul-James White, *m -bigness in compatible families*, preprint, 2010.
- [Wil09] R. Wilson, *The Finite Simple Groups: An Introduction*, Springer, 2009.

E-mail address: `tbl@brandeis.edu`

DEPARTMENT OF MATHEMATICS, BRANDEIS UNIVERSITY, 415 SOUTH STREET MS 050, WALTHAM MA